Non-linear laminar flow of fluid into an open bottom well

S.K. Jain *

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ABSTRACT
In steady state condition, non-linear laminar flow of fluid into an open bottom well just penetrating the semi-infinite porous aquifer is considered. The influence of non-linear flow on discharge and its dependence on related physical quantities is examined. It is found that an open bottom well actually behaves like a hemispherical well, which is an obvious practical phenomenon.

RIASSUNTO
Si considera il flusso laminare non-lineare di un fluido in un pozzo a fondo aperto che penetri un acquifero poroso semi-infinito. Si esamina poi l'influenza del flusso laminare non-lineare sulla discarica e la dependenza da quantità fisiche correlate. Si trova che un pozzo a fondo aperto si comporta come un pozzo emisferico che è un fenomeno pratico ovvio.

1. INTRODUCTION
In the study of flow through porous media, there exists a large number of investigations seeking steady state solutions of Darcy's law.
law with respect to certain applications. One of the most im-
portant applications is the study of flow of fluid from strata into wells.
Many investigators obtained a series of such solutions (cf. [1], [2],
[3]). The well penetrating fully the fluid bearing strata of finite
thickness is termed as fully penetrating well otherwise partially
penetrating well. This is a mathematical concept, a well is always
considered to be partially penetrating. In such cases, the flow in
the region of stratum not penetrated by the well will have an upward
component while the flow in the upper portion of the well will mainly be
tended to bring the fluid into the well, while the flow in the upper portion of the well will mainly be
tended to bring the fluid into the well. The most interesting limiting case of a partially penetrating well is an
open bottom well, which corresponds to a well just tapping a

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stratum. Beside the nature of a well the other important aspect is the
type of fluid which mainly depends on the fluid velocity and the
structural constitution of the porous matrix through which it
flows. On the basis of fluid velocity, however, the flow can be
characterized into three distinct regimes — laminar, nonlinear
laminar and turbulent. The former two types of flow are mainly to
be expected when the fluid velocity is low whereas the latter
regime is only to be expected when the fluid velocity is high. It
does not always purely the natural flow of fluid to be purely
laminar and it appears more desirable to be either nonlinear
laminar or turbulent. Consequently, UCHIDA, 1952, ENGELUND,
1953, ANANDKRISHAN and VARADARAJULU, 1963, WRIGHT, 1968, AHMAD
and SUNADA, 1969, KHAN and RAZA, 1972, JAIN and UPADHYAY, 1976,
UPADHYAY, 1977a, b, and others obtained solutions of certain specific
non-linear laminar flow problems.

In the present paper, we consider non-linear laminar radially
symmetrical flow of an incompressible fluid into an open bottom
well in steady state condition. The influence of non-linear laminar
flow on discharge and its dependance on related physical
quantities is examined. It is found that an open bottom well
actually behaves like a hemispherical well, as in the former case
where the well just penetrates the upper surface of semi-infinite
aquifer, an hemispherical cavity is automatically formed to pro-
provide a surface for discharge. This is an obvious practical pheno-
menon.
2. BASIC EQUATIONS OF FLUID FLOW IN POROUS MEDIA

The Darcy's law governing the laminar flow of fluid in porous media is [2]

\[ v = \frac{K \Delta h}{d_s} \]  

where \( v \), \( K \), and \( \frac{\Delta h}{d_s} \) denote the seepage velocity, seepage coefficient and hydraulic gradient respectively. Flow being in the opposite direction of increasing \( h \).

In case of laminar flow, the total head \( H \) at an infinitely large distance from the axis of the circular perforation of radius \( r \) in the plane impervious boundary of semi-infinite homogeneous aquifer is

\[ H_c = H_0 + \frac{Q}{2\pi r} \]  

where \( Q \) denotes the flow rate and \( H_0 \) the head at the well surface.

Besides, relations [1] and [2], the law for non-linear laminar flow is [2]

\[ \frac{\Delta h}{d_s} = a\rho + b\mu \]  

where \( a \) and \( b \) being constants, which according to Engelund are

\[ a = \frac{2000\mu \rho}{\gamma d} \quad b = \frac{2000\mu}{\gamma d} \]

where \( \rho \) and \( \mu \) being density and viscosity of the fluid respectively and \( d \) the grain size of the medium.
3. STATEMENT OF THE PROBLEM

In steady state condition, we consider radially symmetrical flow of an incompressible fluid into an open bottom well which is a circular perforation with radius $r_w$ of an impervious boundary of semi-infinite porous aquifer (Fig. 1). The aquifer is assumed to be homogeneous and isotropic. The pressure at the contour of well and at the contour of intake are prescribed as $p_w$ and $p_c$ respectively.

Let $r$ be the radial distance measured from the axis of well. It is assumed that the flow is (i) non-linear laminar within the circular zone $r < r_w$, and (ii) laminar in the region $r > r_w$. Let $p_1$ be the pressure at the transition boundary $r = r_w$.

The problem is to examine the influence of non-linear laminar flow on discharge and its dependence on the grain size of the medium and the viscosity of the fluid.
4. Solution

Since \( p = \frac{1}{2} \rho u^2 \), we write (2) as

\[
\beta = \frac{\partial p}{\partial r} = \frac{\partial P}{\partial r} \left( \frac{1}{r^2} \right).
\]

\[
\int \beta dr = \int \frac{\partial P}{\partial r} \left( \frac{1}{r^2} \right) dr
\]

\[
p = C - \frac{\partial P}{\partial r} \left( \frac{1}{r^2} \right).
\]

where \( C \) is the constant of integration determinable by boundary conditions. Thus, equation (7) describes the pressure distribution \( p \) for any arbitrary radius \( r \) (\( r_w < r < r_n \)) in the system for which equation (4) holds good.

Hence, from (7) the expression for pressure distribution at a radial distance \( r \) in the laminar zone \( r < r_w \) is obtainable in the form

\[
p = C - \frac{\partial P}{\partial r} \left( \frac{1}{r^2} \right).
\]

For radially symmetrical flow, equation (1) becomes

\[
\frac{\partial}{\partial r} \left( \frac{\partial P}{\partial r} \right) = \frac{\partial^2 P}{\partial r^2}.
\]

Substituting the value of \( \frac{\partial^2 P}{\partial r^2} \) from (7), equation (8) becomes

\[
v = \frac{C}{r}.
\]

where \( v \) is the seepage velocity at a radial distance \( r \). Thus, \( v \) is independent of the radial \( P \) and \( z \), the vertical coordinate.
Using $z = p$ in equation [9] and then combining it with relation [28], the expression for pressure distribution in the non-linear laminar flow is obtainable in the form:

$$\frac{dz}{dr} = \frac{P}{g} \left( \frac{4q}{x} \right) + \frac{b Q}{16 r^4}, \quad x < \frac{r}{u}, \quad r < r_0, \quad \text{relations [10] and [11].}$$

Integrating [11] and evaluating the constant of integration with the help of boundary conditions

$$z = \left. \frac{b Q}{2} \right|_{r=r_0}$$

we obtain

$$z = \left. \frac{b Q}{2} \right|_{r=r_0} \left[ \frac{4q}{x} \left( \frac{1}{u} - 1 \right) + \frac{b Q}{16 r^4} \right].$$

At the boundary of transition from laminar to non-linear laminar flow, the relation between critical Reynolds's number $\frac{U}{v} = 0.07$ and critical velocity $v_c$ is given by

$$\frac{U}{v} = \left. \frac{1}{v} \right|_{r=r_0} = 0.07.$$

Since at this boundary $\left. \frac{dz}{dr} \right|_{r=r_0}$ given [1] and [3] yield the same value, it follows from [14] that

$$1.07 \times 1 = 1, \quad \text{relations [15].}$$

Using [15] in [10] and then equating it in [10], we get

$$\left( \frac{1}{v} \right)_{r=r_0} = \left[ \frac{1}{x} \left( \frac{1}{u} - 1 \right) + \frac{b Q}{16 r^4} \right].$$
Combining equations \([4 \ a, \ b]\) and \([14]\) with \([16]\), we obtain
\[
\left[ \frac{1}{r_1} + \frac{2}{r_2} \right] + \frac{1}{r_3} = \frac{8000}{-1.5} + 0.0233 \left( \frac{r_1}{r_2} \right)^2 + 0.0467 \left( \frac{r_1}{r_2} \right)^4
\]
where
\[
1 = \sqrt{\frac{r_1 r_2}{r_2}}
\]
If we assume purely laminar flow in the region \(r_1 \ldots\), then the flow rate \(Q_{am}\) is obtained from \([1]\) in the form

\[
Q_{am} = \frac{4 \pi}{3} \left( \frac{r_2}{r_1} \right)^3 \eta^2
\]

Hence from \([14]\) and \([16]\), we obtain the ratio

\[
\frac{Q_{am}}{Q_{am}} = \frac{8560}{8560}
\]

Introducing dimensionless quantity \(X\) and ratio \(\gamma\), we obtain
\[
X = \frac{4 \pi}{3} \left( \frac{r_2}{r_1} \right)^3 \eta^2 \quad \gamma = \frac{r_2}{r_1} \quad \left(1 \ a, \ b\right)
\]

and combining \([16]\) with \([1]\), we obtain an implicit relation

\[
X = \frac{8560}{8560} + 0.0233 \left( \frac{X}{\gamma} \right)^2 + 0.0467 \left( \frac{X}{\gamma} \right)^4
\]

This expression is the same as that obtained by the author \([\text{JAIN, 1981]}\), which describes the non-linear laminar flow into a hemispherical well.
5. Conclusion

It may thus be concluded that an open bottom well actually behaves like a hemispherical well, and in the former case where the well just penetrates the upper surface of a semi-infinite porous medium, a hemispherical cavity is automatically formed to provide a surface for discharge. This is an obvious practical phenomenon, as our conclusion based on theoretical discussion justifies the statement: "...that is a practical case when well just taps the sand, the well surface is really a hemisphere."
REFERENCES


JAIN, S.K., 1981 - Non-linear laminar flow of fluid into a hemispherical well just penetrating the semi-infinite porous aquifer, "Gerlands Beiträge Zur Geophysik".


