On the influence of the vertical density structure on the dynamics of small basins, with specific application to the Adriatic Sea

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Summary. — The importance of a vertical density structure is discussed for what the hydrologic response and circulation pattern of small basins are concerned. The existence of a density stratification is parameterized through a penetration depth of vertical mixing, and different physical situations are analyzed under a specific set of model assumptions and in basins of sizes similar to the Adriatic Sea.

A specific application is made for the Winter hydrological situation in the Adriatic.

Riassunto. — Viene discussa l'importanza di una struttura verticale della densità nel determinare la risposta idrologica ed il campo di circolazione in bacini di piccole dimensioni. La stratificazione verticale della densità é parametrizzata tramite una profondità di penetrazione del mescolamento verticale. Situazioni fisiche diverse vengono analizzate in base ad assunzioni specifiche valide per bacini di dimensioni simili al mare Adriatico, la cui situazione idrologica invernale viene analizzata tramite un'applicazione particolare del modello ed il confronto coi dati sperimentali.

1) Introduction

In this paper, we analyze the dynamical behaviour of small basins, with geometrical scales similar to those of the Adriatic Sea,

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in which a dominant role in determining the hydrographic response and the circulation pattern is played by a complex — and a priori completely arbitrary density structure.

Particular emphasis is given to the importance of the vertical dependence in the density function through the definition of a "penetration depth" D_s, which is the scale depth of vertical change of the density profile. Different values of the penetration depth D_s determine the relative importance of the natural time scales of the motion, and characterize therefore completely different dynamical behaviours of the considered basin. A discussion is given of the assumptions involved in adimensionalizing and approximating the equations of motion, according to the various physical cases defined by different values of the D_s and in the context of a regular expansion procedure in a characteristic small parameter. We then analyze in detail the specific case which idealizes a characteristic Winter hydrological situation of the Adriatic Sea. This situation has been described and modelled mathematically in a previous paper (Hendershott and Rizzoli) [1]. Here we justify through a rigorous procedure the basic physical assumption of the over-mentioned model, deriving again the established fundamental theoretical results which were moreover confirmed by the experimental evidence (Hendershott-Rizzoli; Malanotte Rizzoli) [1,2].

2) The governing equations

We consider a rotating, stratified, incompressible, hydrostatic ("thin" system, aspect ratio of the basin $\lambda = \frac{D_0}{L} \ll 1$) Boussinesq fluid. In basins of the sizes of the Adriatic Sea, the beta effect can be neglected; the Coriolis parameter is therefore assumed constant (f-plane) and given by the average latitude of the considered basin. If, moreover, one makes the assumption that density is linearly related to both temperature and salinity, one can express heat and salt conservation through a single equation for mass conservation.

We follow the traditional approach in ocean circulation modelling of expressing viscous terms through constant eddy viscosity and eddy diffusivity coefficients. Even if these coefficients represent a very rough model of the complex dynamics involved in the representation of the viscous stress terms, this classical line of approach can still give a very considerable insight in the understanding of the dynamics of circulation phenomena and it is widely used in recent and fundamental fluid dynamics studies (Pedlosky) (3,4,5).

In these hypotheses, the dimensionful equations of motion are:

$$
\begin{aligned}
u'_{t'} + u'u'_{x'} + v'u'_{y'} + w'u'_{z'} - fv' &= -p'_{x'}/\varrho_0 + A_v u'_{z'z'} + A_H \nabla_H^2 u' \\
v'_{t'} + u'v'_{x'} + v'v'_{y'} + w'v'_{z'} + fu' &= -p'_{y'}/\varrho_0 + A_v v'_{z'z'} + A_H \nabla_H^2 v' \\
0 &= -p'_{z'}/\varrho - g \qquad [1] \\
u'_{x'} + v'_{y'} + w'_{z'} &= 0 \\
\varrho_{t'} + u'\varrho_{x'} + v'\varrho_{y'} + w'\varrho_{z'} &= K_v \varrho_{z'z'} + K_H \nabla_H^2 \varrho
\end{aligned}
$$

with the usual meaning of symbols. (*)

Here, subscripts indicate partial derivatives, and:

A_v = vertical eddy viscosity coefficient

A_H = horizontal eddy viscosity coefficient

K_v = vertical eddy diffusivity coefficient

K_H = horizontal eddy diffusivity coefficient

ϱ_0 — mean density of the basin

$\nabla_H^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ = horizontal Laplacian

We proceed therefore to adimensionalize the equations through the following scales:

$$(x', y') = X\,(x, y)$$

$$z' = D_0 z$$

with $-d \leqslant z \leqslant 0$ and $d = \text{depth}/D_0$ is the dimensionless variable depth;

$$f = f_0 \Phi = f_0$$

Coriolis parameter;

$$\varrho = \varrho_0 + 10^{-3}\sigma = \varrho_0 + 10^{-3}\,\Delta\sigma \cdot S$$

where σ is nearly the usual σ_t (essentially $\sigma = \sigma_t$ in seas as shallow as the Northern Adriatic), $\Delta\sigma$ is its observed maximum spatial variation and S is the dimensionless density;

$$(u', v') = U\,(u, v)$$

(*) For completeness, a table of all symbols can be found at the end.

with

$$U = g D_0 \, 10^{-3} \, \Delta\sigma / f_0 X \, \varrho_0$$

as given by thermal wind relationship;

$$p' = \bar{p} \, p$$

after subtraction of the hydrostatic pressure due to the mean density ϱ_0, according to the usual convention, and

$$\bar{p} = \varrho_0 \, 10^{-3} \, g \, \Delta \, \sigma \, D_0 \, .$$

The chosen scale for horizontal velocities (that is given by thermal wind relationship) plus a realistic choice for the eddy coefficients in the range of values suitable for basins of sizes comparable to the Adriatic Sea, imply that we deal with circulation models characterized by a geostrophic interior and two, top and bottom, Ekman layers. As in general deep sea circulation models, these Ekman layers will be used in the context of boundary layer approach to satisfy top and bottom boundary conditions for the velocity components; they are, therefore, thin in respect to the total depth.

As a consequence, it can be shown that the vertical velocity w' has a scale depth of vertical change given by the Ekman depth:

$$D_e = \sqrt{A_v / f_0}$$

$$w' = W w$$

with

$$W = \sqrt{A_v / f_0 D_0^2} \cdot D_0 U / X$$

as set by Ekman pumping.

In the case of the Adriatic Sea, the previous scales have the following values:

$$\begin{aligned}
X &= 100\text{-}200 \text{ Km.} \\
D_0 &= 100 \text{ m.} \\
f_0 &= 10^{-4} \text{ sec}^{-1} \\
\Delta\sigma &\simeq 0.5 \text{ gr/cm}^3 \\
U &\simeq 5 \text{ cm/sec} \\
A_v - K_v &- 10^2 \text{cm}^2\text{/sec} \\
A_H - K_H &= 10^6 \text{ cm}^2\text{/sec} \\
D_e &= 10 \text{ m.}
\end{aligned}$$

For what the time scale of the motion is concerned

$$t' = T t$$

we can define three natural time scales of the system:

$$T_{adv} = X/U \text{ advective time scale} \quad (a)$$
$$T_{vd} = D_0^2/K_v \text{ vertical diffusion time scale} \quad (b) \qquad [2]$$
$$T_{Hd} = X^2/K_H \text{ horizontal diffusion time scale} \quad (c)$$

According to the dominant physical mechanism, the time scale will be defined to be [2] (a); (b); (c) respectively.

With the choice of these scales, the adimensionalized equations of motion become:

$$\begin{aligned}
\varepsilon_T u_t + \varepsilon_R(u\, u_x + v\, u_y + \varepsilon_v^{1/2} w\, u_z) - v &= -p_x + \varepsilon_v u_{zz} + \varepsilon_H \nabla_H^2 u \\
\varepsilon_T v_t + \varepsilon_R(u\, v_x + v\, v_y + \varepsilon_v^{1/2} w\, v_z) + u &= -p_y + \varepsilon_v v_{zz} + \varepsilon_H \nabla_H^2 v \\
O &= -p_z - S \\
u_x + v_y + \varepsilon_v^{1/2} w_z &= 0 \\
\Gamma_T S_t + uS_x + vS_y + \varepsilon_v^{1/2} wS_z &= \Gamma_v S_{zz} + \Gamma_H \nabla_H^2 S
\end{aligned} \qquad [3]$$

where

$$\nabla_H^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$$

and the dimensionless parameters are:

$$\varepsilon_T = 1/f_o T$$

with the already mentioned three possible choices for T

$$\varepsilon_R = U/f_o X$$

Rossby number ($\varepsilon_R = \varepsilon_T$ if the time scale is the advective one)

$$\varepsilon_v = A_v/f_o D_o^2$$

vertical Ekman number

$$\varepsilon_H = A_H/f_o X^2$$

Horizontal Ekman number

$$\Gamma_T = T_{adv}/T$$
$$\Gamma_v = T_{adv}/T_{vd}$$
$$\Gamma_H = T_{adv}/T_{Hd}$$

$\Gamma_T = 1$; $\Gamma_T = \Gamma_v$; $\Gamma_T = \Gamma_H$ according to the choice for T.

2.1) *The penetration depth and the natural time scales of the motion.*

To have a suitable parameter directly related to the vertical structure of the density field, we define a scale depth D_s of vertical change of the density profile by supposing horizontal advection and vertical diffusion to be of the same order:

$$D_s^2 = K_v X/U \qquad [4]$$

is therefore the depth of penetration of vertical mixing. With this parametrization, we can express:

$$\Gamma_v = T_{adv}/T_{vd} = D_s^2/D_0^2 \qquad [5]$$
$$\Gamma_H = T_{adv}/T_{Hd} = K_H/XU = K_H\, D_s^2/(K_v X^2)$$

With these definitions, it is intuitive to see how the scale of penetration of vertical mixing D_s is related to the time scale of the motion and, therefore, to the dominating physical mechanism. If $\Gamma_v \ll 1$, this means $D_s \ll D_0$ and $T_{adv} \ll T_{vd}$; the vertical mixing will be limited to a smaller depth as compared to the total one, that is the time necessary for the vertical diffusive process to produce vertical homogeneity in the density field will be much longer than the time required by a water column to make a complete tour in the circulation gyre. If in this case we choose as time scale the smaller T_{adv}, we'll describe the time evolution of the density field as determined primarily by advection. The density field, on the other side, will be stationary (to order zero) when time will have elapsed as to reach the order of magnitude of the T_{vd} scale.

The opposite situation happens if $\Gamma_v \gg 1$, that is $D_s \gg D_0$ or $T_{adv} \gg T_{vd}$. The time necessary for vertical mixing to be complete is now much smaller than the advection time. We'll choose the T_{vd} time scale if we want to describe the details of the time evolution of the vertical mixing process in the density field. If, on the other hand, we limit ourselves to consider motions with time scales of the order of T_{adv} or larger, these motions will see the vertical distribution of the density field as completely homogeneous, and for them the vertical diffusion process will be essentially instantaneous ($T_{vd} = 0$).

The time scale T_{Hd} is the greatest of the three. In the limit of time scales of this order, it can be shown, in the context of the expansion procedure we are exposing in the following, that the density field S has reached the stationary situation, as well as the velocity field. In fact:

$$T_{adv} = \Gamma_H\, T_{Hd}$$

$$T_{vd} = \frac{\Gamma_H}{\Gamma_v}\, T_{Hd}$$

Being Γ_H a small dimensionless parameter (in the case of the Adriatic $\Gamma_H \simeq 2 \cdot 10^{-2}$), $T_{adv} \ll T_{Hd}$. If $\Gamma_v \gg 1$, then $T_{vd} \ll T_{Hd}$. In the opposing limit $\Gamma_v \ll 1$, T_{vd} and T_{Hd} can become of the same order of magnitude, and the choice of either time scale is equivalent to describe the physical process. We therefore limit ourselves to consider the two time scales T_{adv} and T_{vd} in the two extreme cases $\Gamma_v \ll 1$ and $\Gamma_v \gg 1$.

To further specify the physical situation we'll deal with, we make some assumptions on the relative magnitude of the dimensionless small parameters appearing in the momentum equation. For what the time dependence is concerned

$$\varepsilon_T = 1/f_0 T = \varepsilon_R \qquad \text{if } T = T_{adv}$$

$$\varepsilon_T = \Gamma_v \, \varepsilon_R \qquad \text{if } T = T_{vd}$$

Therefore, even with a very small Rossby number, in the case of motions with time scale T_{vd} and $\Gamma_v \gg 1$, the time dependence in the momentum equations might not be negligible.

We put ourselves in the following limit, which is valid for basins similar in size to the Adriatic Sea:

$$\varepsilon_H < \varepsilon_R < \varepsilon_v \ll 1$$

$$\Gamma_H \sim 0\,(\varepsilon_v)$$

With the scales suitable for the Adriatice Sea, in fact,

$$\varepsilon_H = A_H/f_0 X^2 \simeq 10^{-4} \qquad \varepsilon_v = A_v/f_0 D_0^2 \simeq 10^{-2}$$

$$\varepsilon_R = U/f_0 X \simeq 5 \cdot 10^{-3} \qquad \Gamma_H \simeq 2 \cdot 10^{-2}$$

In the momentum equations we therefore neglect the non linear advective terms and the lateral diffusion terms in respect to vertical diffusion terms, keeping for the moment the time dependence which might become important in one specific case, as previously mentioned. Our model equations will be:

$$\begin{aligned}
\varepsilon_T u_t - v &= -p_x + \varepsilon_v u_{zz} \\
\varepsilon_T v_t + u &= -p_y + \varepsilon_v v_{zz} \\
0 &= -p_z - S \\
u_x + v_y + \varepsilon_v^{1/2} w_z &= 0 \\
\Gamma_T S_t + u S_x + v S_y + \varepsilon_v^{1/2} w S_z &= \Gamma_v S_{zz} + \Gamma_H \nabla_H^2 S
\end{aligned} \qquad [6]$$

with:

$\Gamma_T = 1$; $\varepsilon_T = \varepsilon_R$ (negligible time dependence) if $T = T_{adv}$

$\Gamma_T = \Gamma_v$; $\varepsilon_T = \Gamma_v\, \varepsilon_R$ if $T = T_{vd}$.

3) The expansion procedure and the different limiting cases

We choose Γ_H as the small dimensionless parameter characteristic of the system and we take a power of Γ_H, let us say Γ_H^a, as the small parameter of a regular perturbation expansion of all the field functions:

$$\begin{pmatrix} u \\ v \\ w \\ p \\ S \end{pmatrix} = \sum_{n=o}^{\infty} (\Gamma_H^a)^n \begin{pmatrix} u^{(n)} \\ v^{(n)} \\ w^{(n)} \\ p^{(n)} \\ S^{(n)} \end{pmatrix} \qquad [7]$$

In principle, the value of a is completely general. Even without specifying it, we restrict ourselves to the following hypothesis for what orders of magnitude are concerned

$$\Gamma_H^a \sim 0\,(\varepsilon_v^{1/2}) \qquad [8]$$

which implies $0 < a \leqslant 1$ for the basins we are considering, similar in size to the Adriatic Sea.

Therefore, to every order in Γ_H^a, the velocity field can be split into a geostrophic interior plus two top and bottom Ekman layers. The details of Ekman layer analysis will be given in paragraph (4). With this in mind, let us consider separately the two chosen natural time scales and, in the context of each of them, the two limiting cases $\Gamma_v \ll 1$ and $\Gamma_v \gg 1$.

3.1) $T = T_{vd}$

The model equations are:

$$\begin{aligned} \Gamma_v \varepsilon_R u_t - v &= -p_x + \varepsilon_v u_{zz} \\ \Gamma_v \varepsilon_R v_t + u &= -p_y + \varepsilon_v v_{zz} \\ 0 &= -p_z - S \\ u_x + v_y + \varepsilon_v^{1/2} w_z &= 0 \\ \Gamma_v S_t + u S_x + v S_y + \varepsilon_v^{1/2} w S_z &= \Gamma_v S_{zz} + \Gamma_H \nabla_H^2 S \end{aligned} \qquad [9]$$

Many subcases might be distinguished according to the relative magnitude of Γ_v and Γ_H^a. We'll limit ourselves, anyway, to the most significant ones, always in the context of expansion [7].

3.1a) $\Gamma_v \ll 1$

Order zero

$$\begin{aligned} -v^{(0)} &= -p_x{}^{(0)} + \varepsilon_v u_{zz}{}^{(0)} \\ u^{(0)} &= -p_y{}^{(0)} + \varepsilon_v v_{zz}{}^{(0)} \\ S^{(0)} &= -p_z{}^{(0)} \\ u_x{}^{(0)} + v_y{}^{(0)} + \varepsilon_v^{1/2} w_z{}^{(0)} &= 0 \\ u^{(0)} S_x{}^{(0)} + v^{(0)} S_y{}^{(0)} &= 0 \end{aligned}$$

Order $\Gamma_H{}^a$ [10]

$$\begin{aligned} -v^{(1)} &= -p_x{}^{(1)} + \varepsilon_v u_{zz}{}^{(1)} \\ u^{(1)} &= -p_y{}^{(1)} + \varepsilon_v v_{zz}{}^{(1)} \\ S^{(1)} &= -p_z{}^{(1)} \\ u_x{}^{(1)} + v_y{}^{(1)} + \varepsilon_v^{1/2} w_z{}^{(1)} &= 0 \end{aligned}$$

$$\left\{ \begin{array}{l} u^{(0)} S_x{}^{(1)} + v^{(0)} S_y{}^{(1)} + u^{(1)} S_x{}^{(0)} + v^{(1)} S_y{}^{(0)} + \delta\, w^{(0)} S_z{}^{(0)} = 0 \\ \qquad \text{if } \Gamma_v < \Gamma_H^a \\ \nu S_t{}^{(0)} + u^{(0)} S_x{}^{(1)} + v^{(0)} S_y{}^{(1)} + u^{(1)} S_x{}^{(0)} + v^{(1)} S_y{}^{(0)} + \delta\, w^{(0)} S_z{}^{(0)} = \nu S_{zz}{}^{(0)} \\ \qquad \text{if } \Gamma_v \sim \Gamma_H^a \end{array} \right.$$

with $\delta = \varepsilon_v^{1/2} \Gamma_H{}^{-a} \simeq 0(1)$ and $\nu = \Gamma_v \cdot \Gamma_H^a \simeq 0(1)$ according to [8]. In the horizontal momentum plus continuity equations we have kept the z-partial derivatives, weighted by ε_v, to mean that, to both orders, the total velocity field must be split into a geostrophic interior plus two Ekman contributions which decay outside the two Ekman layers. If, in fact, we should consider the Ekman layer equations, we should first stretch the vertical coordinate in system [9] putting: $z = \varepsilon_v^{-1/2} z$ or $\bar{z} = \varepsilon_v^{1/2}(z + d)$ for the top and bottom Ekman layers respectively, and, in the boundary momentum equations, apply the formal expansion procedure in Γ_H^a. The density equations, on the other hand, are the zero and Γ_H^a orders of the expansion of the geostrophic interior density equations. In this case, as discussed in (2-1), $D_s \ll D_o$, and the proper time scale to see any time evolution of the density field to zero order ought to be the smaller T_{adv}. As $T = T_{vd}$, and vertical diffusive processes have a longer time scale, the zero order field $S^{(0)}$ is stationary and is a function of a zero order stream function $\psi^{(0)}$ for the velocity, which can be introduced through the continuity equation of the geostrophic interior ($u_x{}^{(0)} + v_y{}^{(0)} = 0$) that is: $S^{(0)} = f\,(\psi^{(0)})$.

3.1b) $1 \ll \Gamma_v < \varepsilon_R^{-1}$

Order zero

$$\begin{aligned}
-v^{(0)} &= -p_x^{(0)} + \varepsilon_v u_{zz}^{(0)} \\
u^{(0)} &= -p_y^{(0)} + \varepsilon_v v_{zz}^{(0)} \\
S^{(0)} &= -p_z^{(0)} \\
u_x^{(0)} + v_y^{(0)} + \varepsilon_v^{1/2} w_z^{(0)} &= 0 \\
S_t^{(0)} &= S_{zz}^{(0)} \\
\text{and } \gamma &= \Gamma_v \cdot \Gamma_H^a \sim 0(1)
\end{aligned}$$

Order $\Gamma_H{}^a$ [11]

$$\begin{aligned}
-v^{(1)} &= -p_x^{(1)} + \varepsilon_v u_{zz}^{(1)} \\
u^{(1)} &= -p_y^{(1)} + \varepsilon_v v_{zz}^{(1)} \\
S^{(1)} &= -p_z^{(1)} \\
u_x^{(1)} + v_y^{(1)} + \varepsilon_v^{1/2} w_z^{(1)} &= 0 \\
\gamma\, S_t^{(1)} + u^{(0)} S_x^{(0)} + v^{(0)} S_y^{(0)} &= \gamma\, S_{zz}^{(1)} \\
\text{if } \Gamma_v &\sim \Gamma_H^{-a}
\end{aligned}$$

In this case, we are in the situation in which vertical mixing is a very rapid process compared to horizontal advection, and we are describing its time evolution. The zero order density field $S^{(0)}$ obeys the heat equation in the vertical direction, the solutions of which are well known, and should be chosen according to the specific boundary conditions to be applied to S at $z = 0$; $-d$.

3.1c) $\Gamma_v \gg 1$ and $\Gamma_v \geqslant \varepsilon_R^{-1}$

This is the only case in which the time dependence in the momentum equations becomes important. The velocity field is not steady any more, and wave behaviours may well be expected to dominate. The equations for $S^{(0)}$, $S^{(1)}$ remain identical to case (3.1b).

3.2) $T = T_{adv}$

The model equations are:

$$\begin{aligned}
-v &= -p_x + \varepsilon_v u_{zz} \\
u &= -p_y + \varepsilon_v v_{zz} \\
0 &= -p_z - S \qquad [12] \\
u_x + v_y + \varepsilon_v^{1/2} w_z &= 0 \\
S_t + uS_x + vS_y + \varepsilon_v^{1/2} w S_z &= \Gamma_v S_{zz} + \Gamma_H \nabla_H{}^2 S
\end{aligned}$$

Again, in the context of expansion [7], let us consider the two extreme cases of a very small or a very big depth of penetration of vertical mixing.

3·2a) $\Gamma_v \ll 1$

Zero order

$$\begin{aligned}
-v^{(0)} &= -p_x^{(0)} + \varepsilon_v u_{zz}^{(0)} \\
u^{(0)} &= -p_y^{(0)} + \varepsilon_v v_{zz}^{(0)} \\
S^{(0)} &= -p_z^{(0)} \\
u_x^{(0)} + v_y^{(0)} + \varepsilon_v^{1/2} w_z^{(0)} &= 0 \\
S_t^{(0)} + u^{(0)} S_x^{(0)} + v^{(0)} S_y^{(0)} &= 0
\end{aligned} \qquad [13]$$

Order Γ_H^{α}

$$\begin{aligned}
-v^{(1)} &= -p_x^{(1)} + \varepsilon_v u_{zz}^{(1)} \\
u^{(1)} &= -p_y^{(1)} + \varepsilon_v v_{zz}^{(1)} \\
S^{(1)} &= -p_z^{(1)} \\
u_x^{(1)} + v_y^{(1)} + \varepsilon_v^{1/2} w_z^{(1)} &= 0
\end{aligned}$$

$$S_t^{(1)} + u^{(0)} S_x^{(1)} + v^{(0)} S_y^{(1)} + u^{(1)} S_x^{(0)} + v^{(1)} S_y^{(0)} + \delta\, w^{(0)} S_z^{(0)} = \begin{cases} 0 & \text{if } \Gamma_v \ll \Gamma_H^{\alpha} \\ \nu\, S_{zz}^{(0)} & \text{if } \Gamma_v \sim \Gamma_H^{\alpha} \end{cases}$$

and $\nu = \Gamma_v \cdot \Gamma_H^{\alpha} \sim 0(1)$

Again $\delta = \varepsilon_v^{1/2}\, \Gamma_H^{-\alpha} \sim 0(1)$ according to [8]. Being $\Gamma_v \ll 1$, as previously noticed, vertical diffusion is slow compared to horizontal advection. With a time scale characteristic of this latter process, the time evolution of the density field to zero order is determined only by advection.

3-2b) $\Gamma_v \gg 1$

Among the various possible cases, we restrict ourselves to consider only the one $\Gamma_v \sim \Gamma_H^{-\alpha}$ as the most significant.

Order zero

$$\begin{aligned}
-v^{(0)} &= -p_x^{(0)} + \varepsilon_v u_{zz}^{(0)} \\
u^{(0)} &= -p_y^{(0)} + \varepsilon_v v_{zz}^{(0)} \\
S^{(0)} &= -p_z^{(0)} \\
u_x^{(0)} + v_y^{(0)} + \varepsilon_v^{1/2} w_z^{(0)} &= 0 \\
S_{zz}^{(0)} &= 0
\end{aligned} \qquad [14]$$

Order Γ_H^{α}

$$\begin{aligned}
-v^{(1)} &= -p_x^{(1)} + \varepsilon_v u_{zz}^{(1)} \\
u^{(1)} &= -p_y^{(1)} + \varepsilon_v v_{zz}^{(1)} \\
S^{(1)} &= -p_z^{(1)} \\
u_x^{(1)} + v_y^{(1)} + \varepsilon_v^{1/2} w_z^{(1)} &= 0 \\
S_t^{(0)} + u^{(0)} S_x^{(0)} + v^{(0)} S_y^{(0)} &= \gamma\, S_{zz}^{(1)}
\end{aligned}$$

In this case, we consider motions with time scales of the order of T_{adv} or larger; for these motions, the vertical diffusion is essentially instantaneous.

4) The winter phenomenology of the Adriatic sea and the relative mathematical model

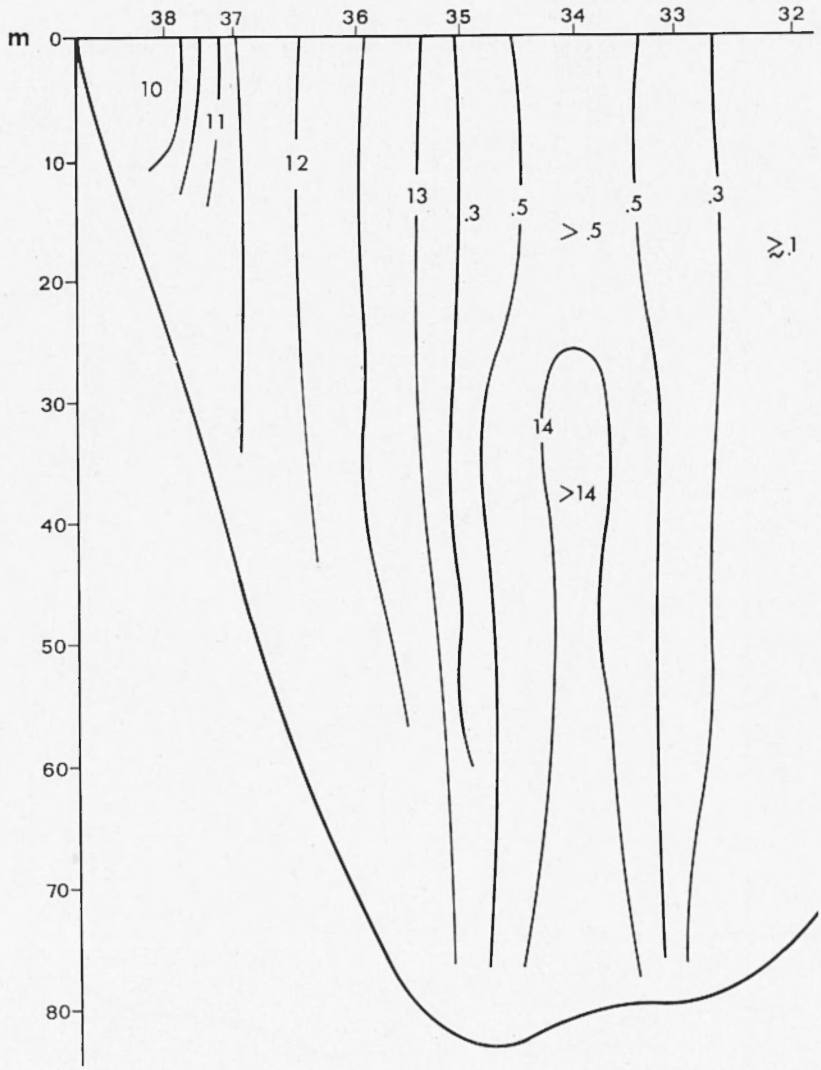
In the previous paragraph we have examined the detailed form of the equations of motion for each of the intrinsic time scales T_{adv} and T_{vd}, and in the extreme, opposite cases of a very small or a very big penetration depth D_s. We have not tried to give an analytic solution to each specific case — even when the form of the equations allowed it — because every system must be viewed in the context of the physical problem under consideration, with the suitable and properly chosen boundary conditions. In this paragraph we are concerned with the Winter hydrological situation in the Adriatic sea, discussed and modelled in the previously mentioned papers (see references).

For basins of the shape and dimensions of the Adriatic, the proper choice and values of the geometrical length scales and of the various parameters has been already pointed out in paragraph (2).

In considering now specifically the Winter situation, we'll notice that, in most of the Winter hidrological data at disposal, the fields of temperature, salinity, and hence density, are vertically mixed to essential homogeneity. This is, in particular, the case of the two Winter oceanographic campaigns the experimental results of which we'll compare with the theoretical predictions, that is the period January-February 1966 (Trotti 1970) [6] and January-February 1972 (P. Malanotte Rizzoli, in press) [2].

In both these periods, the vertical maps of temperature, salinity, density show an essentially complete vertical mixing, which, in the Southern part of the basin, extends to more than 200 m. depth (Fig. 1). Only a narrow boundary strip adjacent to the italian coastline is excluded from this situation of vertical homogeneity. In it, the river outflow — concentrated along the North-Western Italian shoreline — causes the persistence of a noticeable vertical stratification persisting also in Winter time.

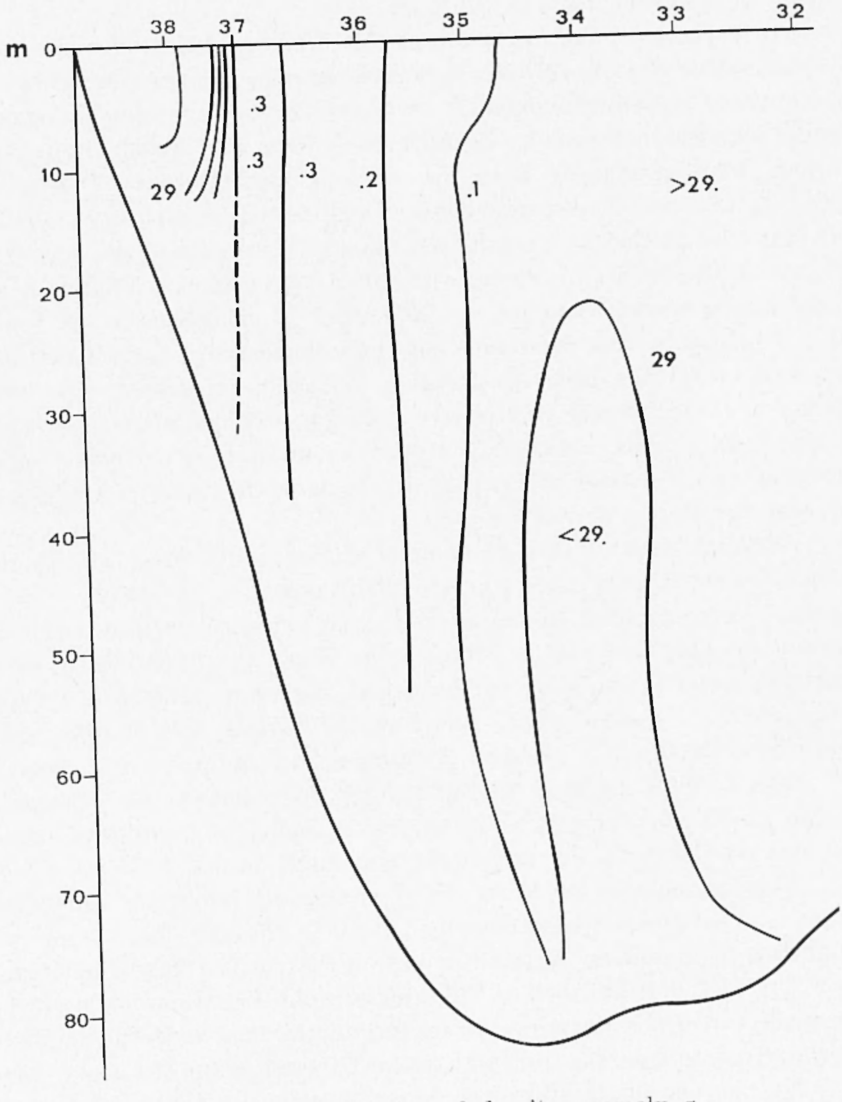
In the two overmentioned Winters, moreover, there is the formation of a pool of water of particularly high density ($\sigma_t > 29.4$)



a) Vertical distribution of temperature in °C.

Fig. 1 a-b-c – P. Malanotte Rizzoli – Cruise January-February 1972 – Vertical maps relative to a cross-section of the Adriatic from Porto Civitanova to the Isola Grossa

b) Vertical distribution of salinity in ‰.



c) Vertical distribution of density anomaly σ_t

in the northernmost part of the basin (Fig. 2). This pool protrudes southward with a characteristic "tail" along the western side of the Adriatic, following isobath contours.

In both cases, the formation of this dense pool can be connected (Hendershott-Rizzoli, 1976; P. Malanotte Rizzoli, in press) [1,2] with two exceptional episodes, anomalous in respect to average meteorological conditions, characterized by the outbreak of dry, cold air of continental origin, blowing directly onto the Adriatic from Euroasiatic North-Eastern regions. These episodes — which lasted in both cases about 15 days — produced exceptionally intense thermal and evaporative fluxes at the air-sea interface, with, as a consequence, complete vertical mixing of water columns and the formation of the dense water pool.

To describe and reproduce this hydrological situation, a mathematical model has been constructed, vertically integrated over the water depth, exploring the relative importance which air-sea thermal fluxes, wind stress, coastal river outflows and water exchange with the Southern Adriatic do have in determining the lields of transport stream function ψ and density S.

Without entering into the details of the model, given elsewhere (Hendershott-Rizzoli, 1976) [1] we'll point out that its primary idealization, corresponding to the experimental evidence, is that vertical mixing of heat and salt is complete for what the dynamics of circulation and the evolution of the density field are concerned. This fundamental hypothesis led, therefore, to assume the density field depending essentially only upon the horizontal coordinates $S = S(x,y)$.

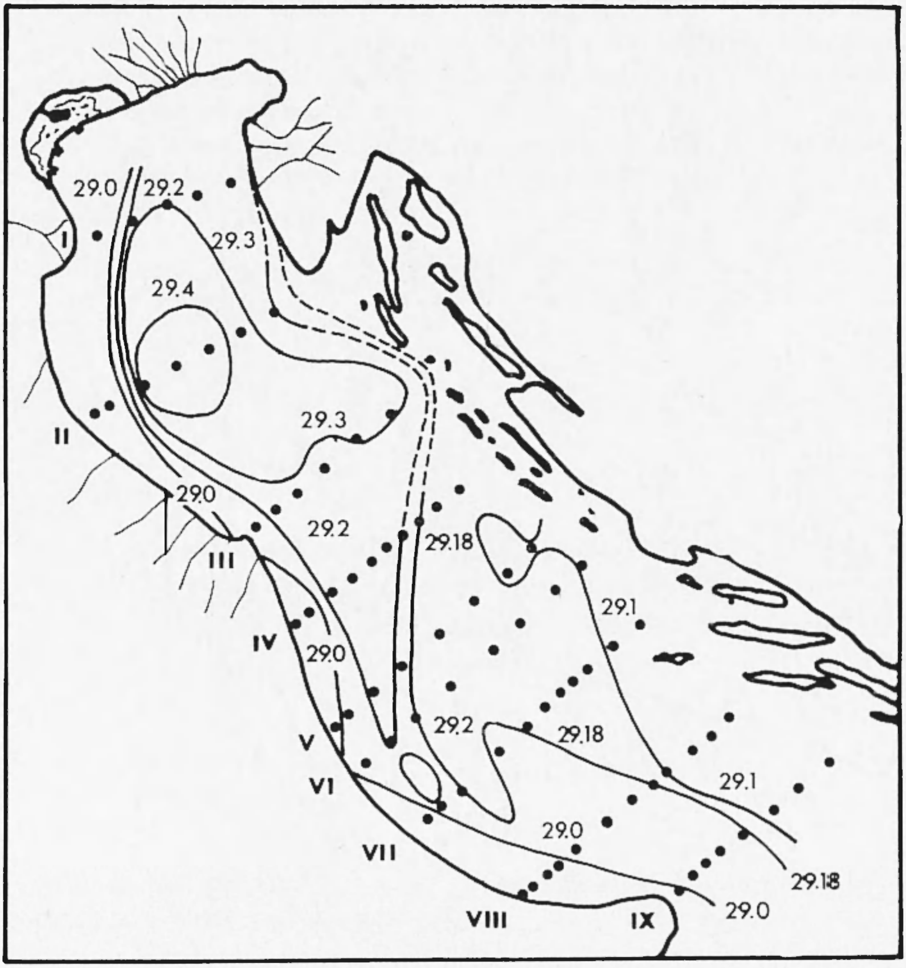
The dependance of S on the vertical coordinate z was assumed to be exprimible through additive terms negligible, in order of magnitude, relatively to the horizontal, dominant term.

This dependance of S on z was considered important only for what air-sea fluxes were concerned, that is through the boundary condition at the air-sea interface $z = 0$ on the vertical flux of density.

This assumption allowed to integrate the equations of motion vertically, over the water depth, leading to the final model equations in the two functions, the transport stream function ψ and the density S.

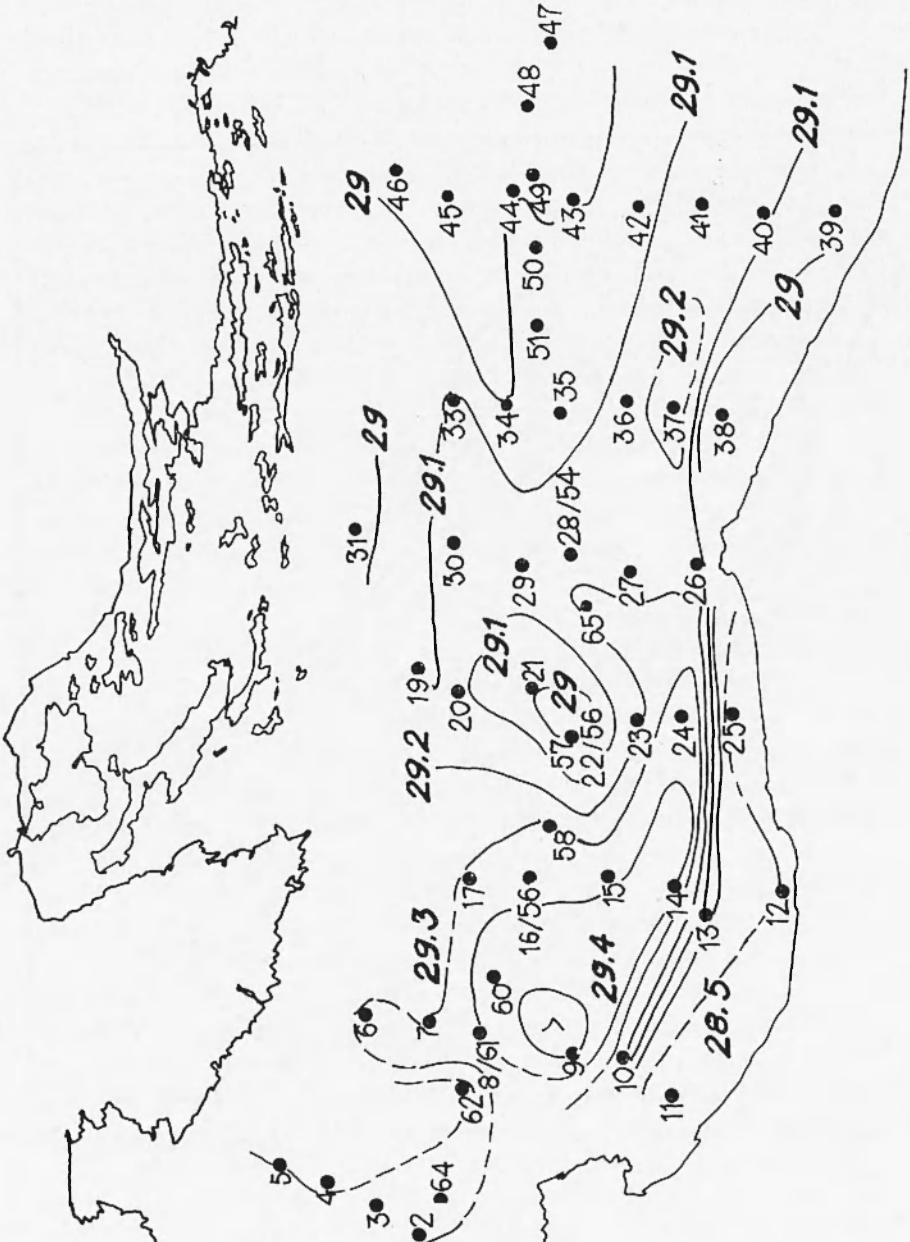
No rigorous justification was, anyway, given for this basic hypothesis, and the purpose of the following treatment is, as previously mentioned, to prove its validity, deriving it through a formal analytical procedure.

In the context of the development made in this paper, which through the expansion [7] has led to distinguish the various cases



(a) Surface distribution of density anomaly for the cruise January-February 1966

Fig. 2 a-b



(b) Surface distribution of density anomaly for the cruise January-February 1972.

examined in paragraph (3), we are in the situation in which $T_{vd} \ll T_{adv}$ and $\Gamma_v \gg 1$.

The vertical diffusion time scale is very small; vertical mixing can be considered instantaneous ($T_{vd} \sim 0$) when dealing with phenomena in which order of magnitude variations in the circulation and density fields occur over times of the order or larger than the advective time scale. We are therefore in the case [3,2b] described by the set of equations [14].

To determine completely the solution to zero order, it is sufficient to couple the zero order system with the order Γ_H^a density equation, getting:

$$\begin{aligned} -v^{(0)} &= -p_x{}^{(0)} + \varepsilon_v u_{zz}{}^{(0)} \\ u^{(0)} &= -p_y{}^{(0)} + \varepsilon_v v_{zz}{}^{(0)} \\ S^{(0)} &= -p_z{}^{(0)} \\ u_x{}^{(0)} + v_y{}^{(0)} + \varepsilon_v{}^{1/2} w_z{}^{(0)} &= 0 \\ S_{zz}{}^{(0)} &= 0 \end{aligned} \qquad [15]$$

and

$$S_t{}^{(0)} + u^{(0)} S_x{}^{(0)} + v^{(0)} S_y{}^{(0)} = \gamma\, S_{zz}{}^{(1)}$$

As previously pointed out, this notation means that two Ekman layers are coupled to a geostrophic interior. In fact, applying the expansion [7] to the interior general equations, the zero order system is:

$$\begin{aligned} v^{(0)} &= p_x{}^{(0)} \\ u^{(0)} &= -p_y{}^{(0)} \\ p_z{}^{(0)} &= -S^{(0)} \\ u_x{}^{(0)} + v_y{}^{(0)} &= 0 \\ S_{zz}{}^{(0)} &= 0 \end{aligned}$$

In analyzing the Ekman layers, one must stretch the vertical coordinate according to $\bar{z} = \varepsilon_v^{-1/2} z$ or $\bar{z} = \varepsilon_v^{-1/2} (z + d)$ for top and bottom Ekman layers respectively. Then, always in the context of expansion [7], the Ekman zero order system is:

$$\begin{aligned} -v^{(0)} &= -p_x{}^{(0)} + u_{\bar{z}\bar{z}}{}^{(0)} \\ u^{(0)} &= -p_y{}^{(0)} + v_{\bar{z}\bar{z}}{}^{(0)} \\ p_{\bar{z}}{}^{(0)} &= 0 \\ u_x{}^{(0)} + v_y{}^{(0)} + w_{\bar{z}}{}^{(0)} &= 0 \\ S_{\bar{z}\bar{z}}{}^{(0)} &= 0 \end{aligned}$$

Following the usual procedure of splitting the velocity components into an interior (geostrophic) part plus an Ekman layer contribution which must decay outside the layer itself, it can esaily be found that the Ekman velocities are:

$$u_E^{(0)} = e^{z/\sqrt{2}} \left(b \sin \left(\frac{z}{\sqrt{2}}\right) + c \cos \left(\frac{z}{\sqrt{2}}\right)\right);$$

$$v_E^{(0)} = e^{z/\sqrt{2}} \left(c \sin \left(\frac{z}{\sqrt{2}}\right) - b \cos \left(\frac{z}{\sqrt{2}}\right)\right)$$

The integration constants b and c are determined applying the boundary conditions which, for the top and bottom layers, are respectively:

$$\left.\begin{aligned} \partial u_{tot}^{(0)} / \partial z &= \partial (u_{interior}^{(0)} + u_E^{(0)}) / \partial z = \tau_W^x \\ \partial v_{tot}^{(0)} / \partial z &= \partial (v_{interior}^{(0)} + v_E^{(0)}) / \partial z = \tau_W^y \end{aligned}\right\} \text{ at } z = 0$$

where τ_W^x, τ_W^y are the dimensionless wind stress components at sea surface $z = 0$, and

$$u_{tot}^{(0)} = u_{interior}^{(0)} + u_E^{(0)} = 0; \qquad v_{tot}^{(0)} = v_{interior}^{(0)} + v_E^{(0)} = 0$$

at $z = -d$, no slip condition at the sea bottom. The vertical velocity component $w_{tot}^{(0)} = w_{interior}^{(0)} + w_E^{(0)}$ can be easily found from the continuity equation and top and bottom boundary conditions, in our case $w_{total}^{(0)} = 0$ at $z = 0, -d$. The zero order velocity field is therefore completely determined. The most important point to be noticed, anyway, is that the Ekman layers are boundary layers for what momentum is concerned but not for the density and pressure fields. Apart from the physical consideration that the Ekman layer is characteristically the one in which vertical diffusion of momentum is the dominant process, this can be rigorously shown considering the Ekman layer form of the density and pressure equations. From these, as any Ekman correction to the interior fields must decay outside the layers themselves, it is evident that these corrections must be identically zero. Therefore, the interior pressure and density determine the total fields of these quantities.

From the zero order equation $S_{zz}^{(0)} = 0$,

$$S^{(0)} = A(x,y,t)\, z + B(x,y,t) \tag{16}$$

The proper surface and bottom boundary conditions for the density are:

$$\Gamma_v S_z = Q \quad \text{at } z = 0; \qquad S_z = 0 \quad \text{at } z = -d$$

At $z = 0$, the air-sea interface, the source function Q includes all heat and water vapor fluxes which produce density changes through cooling and evaporation. At $z = -d$, no density flux is required through the sea bottom. In terms of the expansion [7], this means:

$$\Gamma_v S_z{}^{(0)} + \Gamma_v \Gamma_H^a S_z{}^{(1)} + \ldots = \Gamma_v S_z{}^{(0)} + \gamma\, S_z{}^{(1)} + 0\,(\Gamma_H^a) = Q$$
$$\text{at } z = 0 \quad ; \quad S_z{}^{(0)} = S_z{}^{(1)} = \ldots = 0 \qquad \text{at } z = -d$$

From the bottom boundary condition, it follows:

$$S^{(0)} = S^{(0)}\,(x,y,t) \qquad [17]$$

The zero order density is not vertically dependent. From [17] and the surface boundary condition, it follows that the proper choice for the latter is:

$$\gamma\, S_z{}^{(1)} = Q \qquad \text{at } z = 0 \qquad [18]$$

It is therefore rigorously shown that, under the set of assumptions defining our physical system, the zero order density depends only on horizontal coordinates. The zero order pressure equation can hence be integrated to give:

$$p^{(0)} = \pi\,(x,y,t) - S^{(0)}\,(x,y,t) \cdot z \qquad [19]$$

where $\pi\,(x,y,t)$ is a surface pressure field.

The Γ_H^a order equation for the density can be written as:

$$S_t{}^{(0)} + J\,(\psi^{(0)}, S^{(0)}) = \gamma\, S_{zz}{}^{(1)} \qquad [20]$$

where $u^{(0)} = -\psi_y{}^{(0)}$; $v^{(0)} = +\psi_x{}^{(0)}$ define a zero-order stream function through the continuity equation, and $J\,(a,b) = \partial\, a/\partial\, x \times \partial\, b/\partial\, y - \partial\, a/\partial\, y \times \partial\, b/\partial\, x$ is the Jacobian of the fields a,b.

If $J\,(\psi^{(0)}, S^{(0)}) = 0$ — which implies $\psi^{(0)} = f\,(S^{(0)})$ — and if the source function Q is only a function of time, the density equation [20], integrated from top to bottom, admits the exact solution:

$$S^{(0)} = S_{in}{}^{(0)} + \frac{1}{d}\int_0^t Q\,(t')\,\mathrm{d}t' \qquad [21]$$

and $S_{in}{}^{(0)}$ is the density field at $t = 0$.

The momentum equations previously discussed, if integrated from top to bottom and properly treated in terms of a transport stream function, produce, together with the density equation [20] also integrated vertically, the model equations of the previously mentioned reference, of which equation [21] was a particular solution, confirmed by comparison with the experimental data.

The only difference is that the density model equation of Hendershott-Rizzoli maintained a horizontal diffusion term $\Gamma_H \nabla_H^2 S$, which, in the treatment through the expansion procedure [7], will appear only at higher orders. We must point out, anyway, that the analysis here made is rigorously valid only in the interior of the basin, away from side wall boundaries, where horizontal density gradients, and therefore horizontal diffusion terms, become much more important, and can be inferred to be not negligible. Near these side walls, this expansion procedure in terms of the small parameter Γ_H ought to be inserted in the context of a boundary layer analysis, emphasizing the importance of the diffusion processes responsible for the mixing of the coastal fresh water with the denser one of the interior, and therefore bringing to zero order the horizontal diffusive term.

In the mathematical model previously mentioned, no distinction was made between the interior and the side wall boundary, the model describing the evolution of circulation and density fields in the whole of the basin. The diffusive term, even weighted by Γ_H, was therefore necessary to allow for the mixing of coastal and interior waters, and appeared to be of fundamental importance in determining the time evolution of the properties of the basin.

Even with these points in mind, the preceding analysis shows that, for a sea like the Adriatic and under the physical assumptions apt to model a Winter characteristic situation, the main interior part away from a side wall coastal strip obeys the set of equations [15], leading to the detailed solutions previously outlined.

5) Comparison with experimental results

Having rigorously justified the basic assumption $S = S(x,y)$ of the model, numerical experiments have been carried out; the results of a typical computation are shown in Fig. 3 and 4.

In them, the evolution of the transport stream function ψ and density field S are shown at successive steps of the time integration.

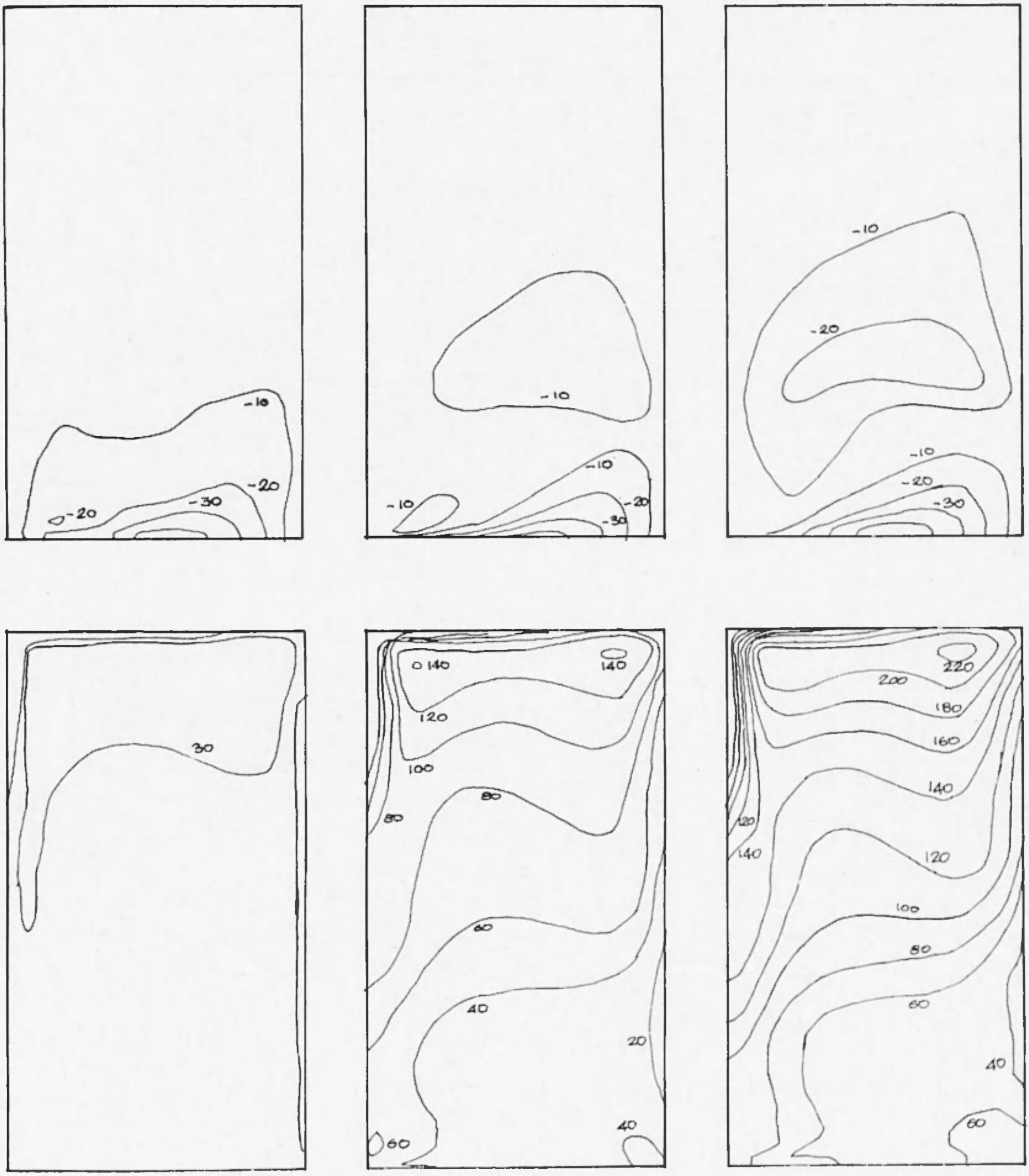


Fig. 3 – Time evolution of transport stream function (a1-a2-a3) and density (b1-b2-b3) both in dimensionless units, multiplied through 100.
Stream function unit: 10^6 m³/sec. Density units: = 29.0 + $\Delta\,\sigma\,S$ with = 0.5
a1 – b1 = time step 5, corresponding to 3.46 days – a2 – b2 = time step 15, corresponding to 10.38 days – a3 – b3 = time step 25, corresponding to 17.38 days

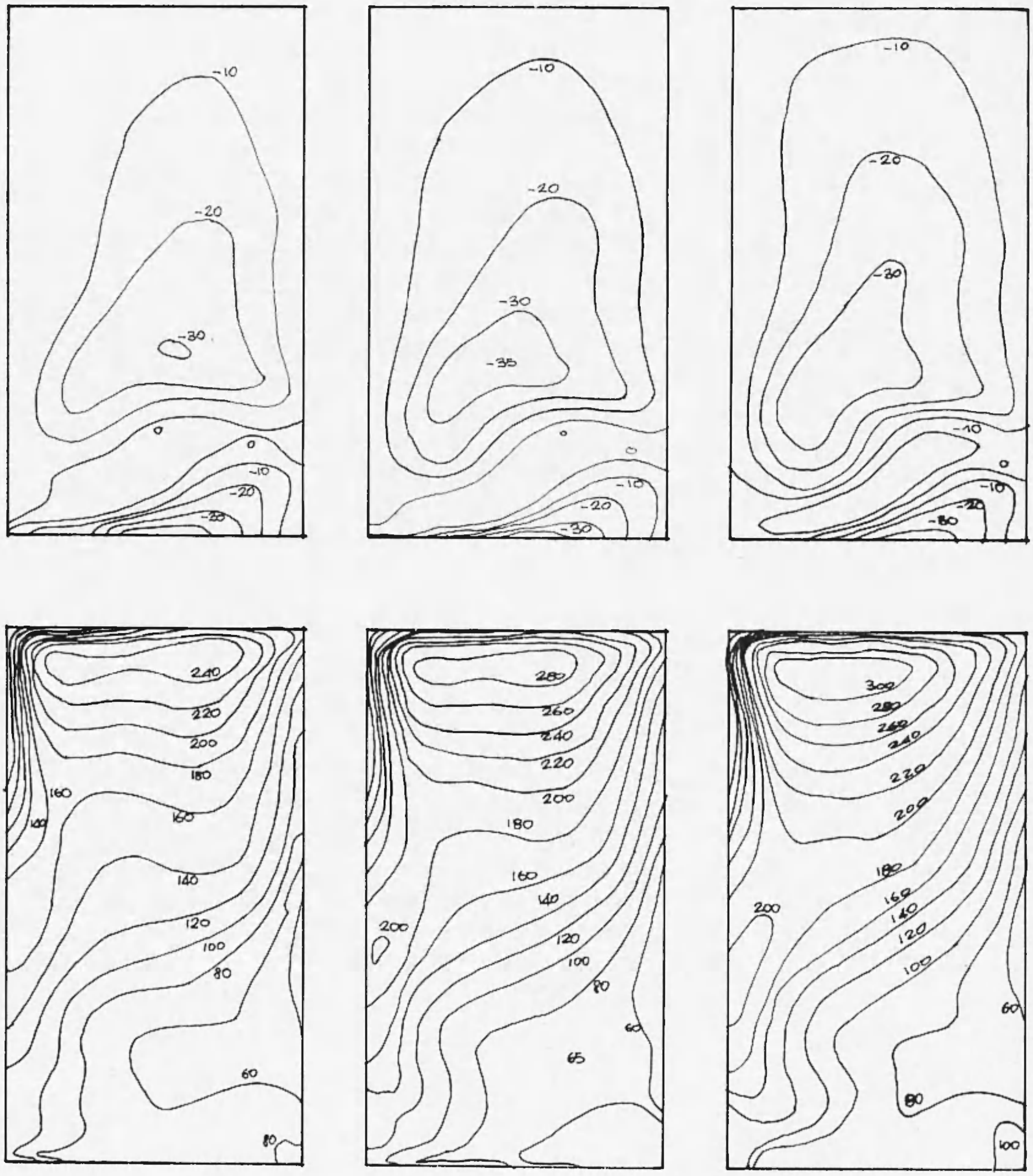
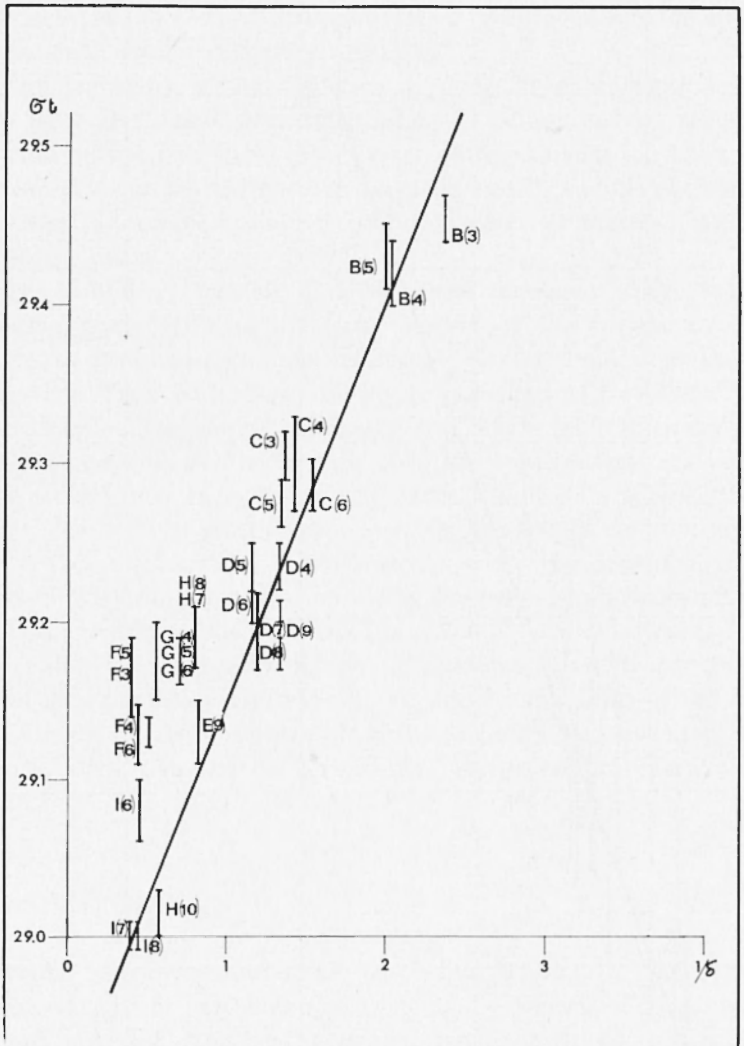
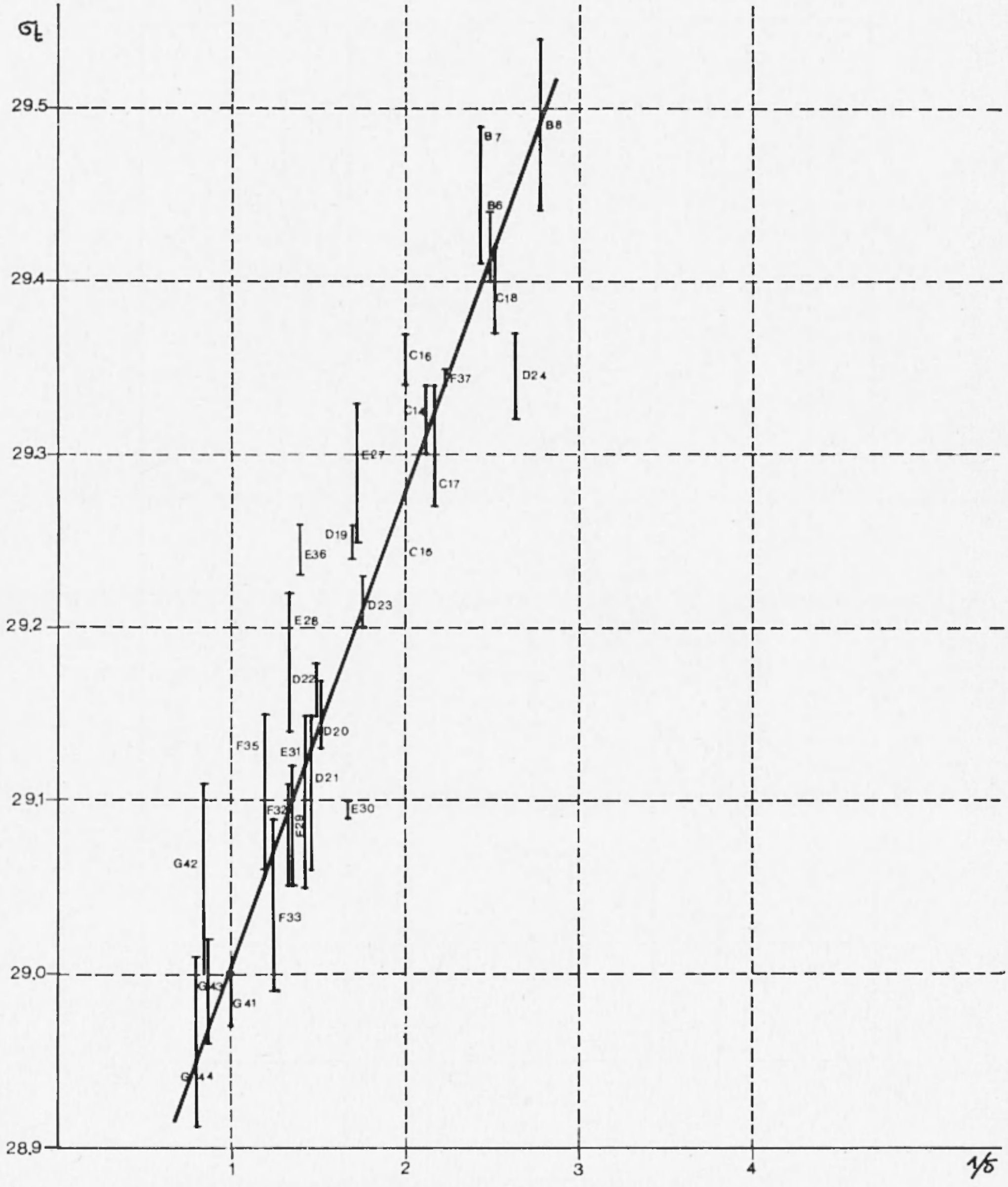


Fig. 4 - As in Fig. (3), transport stream function (a1-a2-a3) and density (b1-b2-b3).
a1 - b1 = time step 35, corresponding to 24.22 days
a2 - b2 = time step 45, corresponding to 31.14 days
a3 - b3 = time step 55, corresponding to 38.10 days



(a)

Fig. 5 a-b – σ_t as plotted versus dimensionless inverse depth for the cruise of Winter 1966 (a) and Winter 1972 (b).



σ_t
29.5
29.4
29.3
29,2
29,1
29,0
28,9
1
2
3
4
$1/\delta$
B8
B7
B6
C18
C16
F37
D24
C14
E27
C17
D19
E36
C15
D23
E28
D22
D20
F35
E31
D21
F32
E29
E30
G42
F33
G43
G41
G44

(b)

It is evident the formation of the dense water pool centered in the Northern part of the basin, stretching southward in the characteristic tail which follows the isobath contours, in very good agreement with experimental results (compare with Fig. 2).

The transport stream function ψ shows a characteristic circulation gyre of thermohaline origin, that is determined by the strong gradients between the light water river outflows along the boundaries and the dense water of the interior. This gyre is completely indipendent from the forcing boundary condition given at the open southern end of the basin.

The analytical solution [21] has been, moreover, compared vith the experimental data at disposal. In Fig. 5 the results are given relative to the oceanographic cruises of Winter 1966 and 1972 respectively. They both show the linear dependance of σ_t from the inverse dimensionless depth $1/d = D_0/D = 100$ m.$/D$. Excluded from the linear dependance region are only the undercoastal stations, in the Western strip adjacent to the Italian shoreline, where, also in Winter, the vertical stratification persists because of river water outflows. This means that, in this coastal strip, the dependance from the vertical coordinate z must appear in the zero order density function $S^{(0)}$, and not only in the first order function $S^{(1)}$ as it has been proved to be valid for the interior region. Therefore, this strip can be considered as a boundary layer in relationship to the whole of the basin, and as such, it can be treated with the techniques of boundary layer analysis, always in the context of the expansion procedure defined by [7].

6) Conclusions

Under the definition of a penetration depth, we have pointed out the variety of mathematical systems — corresponding to widely different physical situations — which can arise in the context of a rigorous formal procedure. We have therefore applied this procedure to the specific case of a Winter hydrologic situation in the Adriatic Sea, showing how the density field can be assumed, to a zero order approximation, to be vertically homogeneous, and re-deriving a set of solutions modelled in a previous paper.

For this study to be complete, a boundary layer analysis must be added to take into account the modifications of the model equations describing the interior fields, when approaching a side wall coastal boundary. In it, horizontal diffusion processes — of higher order in the interior — will become important in producing the local dynamics of mixing and in influencing as a consequence, the evolution itself of the interior fields.

Studies in this direction have been undertaken, and considerable progress is being made.

TABLE OF SYMBOLS

A_v = vertical eddy viscosity coefficient
A_H = horizontal eddy viscosity coefficient
K_v = vertical eddy diffusivity coefficient
K_H = horizontal eddy diffusivity coefficient
X = horizontal length scale
D_o = vertical length scale
f = Coriolis parameter
ϱ_o = mean density of the basin
U = velocity scale
$\Delta\sigma$ = scale for horizontal variations of density anomaly σ_t
D_e = $\sqrt{A_v/f_o}$ Ekman depth
T_{adv} = X/U advective time scale
T_{vd} = D_o^2/K_v vertical diffusion time scale
T_{Hd} = X^2/K_H horizontal diffusion time scale
∇_H^2 = $\partial^2/\partial x^2 + \partial^2/\partial y^2$ horizontal Laplacian
ε_T = $1/f_o T$
ε_R = $U/f_o X$ Rossby number
ε_v = $A_v/f_o D_o^2$ vertical Ekman number
D_s^2 = $K_v X/U$ penetration depth
Γ_v = $T_{adv}/T_{vd} = D_s^2/D_o^2$
Γ_H = $T_{adv}/T_{Hd} = K_H/X \cdot U$
ε_H = $A_H/f_o X^2$ horizontal Ekman number
d = D/D_o dimensionless depth

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