Non-linear laminar flow into eccentrically placed well

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SUMMARY. — In steady state condition, non-linear laminar flow of fluid into an eccentrically placed well is considered. Its influence on the discharge and the dependence on related physical quantities is investigated. It is observed that as the well approaches towards the contour of intake, the discharge increases, which is an obvious result consistent with that obtained by Polubarinova-Kochina in case of laminar flow. As a particular case, result for concentric well has also been deduced.

INTRODUCTION

The intricacy in the nature of porous media does not always justify the natural flow of fluid through it to be purely laminar. However, it appears more justifiable to consider the flow through porous media to be either non-linear laminar or turbulent (*). Consequently Jain and Upadhyay (*), Elenbaas and Katz (*), Engelund (*) obtained specific solutions of some non-linear laminar and turbulent flow problems.

In the present paper, we consider the non-linear laminar steady state flow of fluid into an eccentrically placed well fully penetrating...
the porous aquifer. It is found that the flow pattern is characterised by two different zones, in which discharge exhibits opposite character as regards its dependence on grain size of the medium, viscosity of the fluid and radius of the well. Further, it is observed that as the well approaches the contour of intake, the discharge increases abruptly as compared to that into a concentrically placed well, which is obvious from physical considerations.

The results for a concentric well have been deduced and compared with those obtained by Upadhyay. (5)

2. - EQUATIONS OF FLUID FLOW IN POROUS MEDIUM

The Darcy's law governing the laminar flow of fluid in porous media is

\[ v = k \frac{d\lambda}{ds} \]  

where \( v \), \( k \) and \( \frac{d\lambda}{ds} \) denote the seepage velocity, seepage coefficient and hydraulic gradient respectively; flow being in the opposite direction of increasing \( h \).

In case of an eccentrically placed circular well fully penetrating the cylindrical stratum (radius \( R \)) of unit thickness, the pressure \( p \) at any point with complex coordinate \( z \) is obtained in the form (6)

\[ p = \frac{Q\mu}{2\pi R} \log \left( \frac{R (|z| - 2z_i)}{(R - 2z_i)} \right) + C \]

where \( Q \), \( k \) and \( \mu \) represent flow rate, permeability of the medium and viscosity of the fluid respectively; \( z_i \) denotes the centre of well and \( z_i \) is the corresponding inverse point. The constant \( C \) is to be determined by boundary conditions.

Besides relations [1] and [2], the law for non-linear laminar flow is (4)

\[ \frac{d\lambda}{ds} = \alpha + \beta v^2 \]  

where \( \alpha \) and \( \beta \) are constants. According to Engelund (5)
3. - Formulation of the problem

In steady state condition, we consider the flow of fluid into an uncased circular cylindrical well of radius \( r_e \) eccentrically established at a distance \( R_i \) from the centre of the contour of intake. It is assumed that the well is completely penetrating the porous aquifer of thickness \( T \). The aquifer is considered to be homogeneous and isotropic bounded by horizontal impervious layers. The pressure at the contour of well and at the contour of intake are prescribed as \( p_w \) and \( p_c \) respectively.

Let \( r \) be the radial distance measured from the axis of intake.

As the effect of non-linear laminar or turbulent flow is observed to be appreciable even if such flow is restricted to a comparatively narrow zone \((\varepsilon)\), we consider the flow to be non-linear laminar within a narrow cylindrical zone of radius \( r_t \) surrounding the well and laminar beyond this zone. Let the pressure at the transition boundary be \( p_t \) [Fig. 1].

\[
a = \frac{2000}{\rho d^3} \quad [4]_1 \\
b = \frac{25}{\rho d} \quad [4]_2
\]

\( \rho \) and \( d \) being density of the fluid and grain size of the medium.

\[ (\alpha) = (\alpha) = R_i \]
Therefore, expression for pressure distribution in the laminar zone
(cf. [2]) takes the form

\[ p = \frac{\mu}{2\pi \rho \gamma} \log \frac{R (r - R_i)}{R^2 - R_i^2} + C \]  \[5\]

The problem is to examine the influence of non-linear laminar
flow on discharge of fluid and its dependence on the related physical
quantities.

4. - Solution

As in the vicinity of well the lines of equal pressure are closed to
circles, therefore, we assume the contour of well as one of the isobars
close to the circle of radius \(r_w\) [4]. Along the boundary of transition,
which is close to the contour of well, pressure \(p_t\) may be obtained from
[5] by using the boundary conditions

\[ p = p_t \text{ at } r = R, \]

\[ p = p_t \text{ at } r = R_i + r_t, \quad (r_t < R_i < R). \]  \[6\]

Hence

\[ p_t = p_t + \frac{Q\mu}{2\pi \rho \gamma} \log \frac{R}{R^2 - R_i^2} \]  \[7\]

Since \(p = \rho g h\), pressure distribution in the non-linear laminar
zone is obtainable from [3] as

\[ \frac{1}{\rho g} \frac{\partial p}{\partial x} = ax + bx^2 \]  \[8\]

In general, discharge \(Q\) from any cylindrical surface of radius \(r\)
and height \(T\) is

\[ Q = 2 \pi r T \]  \[9\]

Consequently, in this situation

\[ r = R_i + \lambda, \quad r_w < \lambda < r_t \]  \[10\]
it follows from [8], [9] and [10] that

\[
\frac{dp}{p} = \frac{R_i + r_i}{R_i + r_c} \left( \frac{\alpha f_T}{2 R_i T} + \frac{b Q^2}{4 \pi r^2 (R_i + r)} \right) dr
\]  

[11]

i.e. \( p_c = p_0 + \alpha f_T \left( \frac{R_i + r_i}{R_i + r_c} \right) \frac{b Q^2}{4 \pi r^2 (R_i + r)} \) \[12\]

At the boundary of transition from laminar to non-linear laminar flow, the relation between critical Reynold's number \( \xi_c = 0.07 \) and critical velocity \( v_c \) is given by (9)

\[
v_c = \frac{Q}{2 \pi r_c T} - \xi \frac{a}{b},
\]

where \( \frac{d v}{d \xi} \) as given by [1] and [3] yield the same value. Accordingly

\[
\frac{\xi_c}{v_c} = a v_c (1 + b \frac{v_c}{v_c}),
\]

or, \( \frac{1}{b} = 1.07 a \).

Since \( k = \frac{5 \alpha f_T}{\mu} \), it follows that

\[
\frac{\mu}{k_0} = 1.07 agg
\]

[14]

Using [14] in [7] and then comparing with [12], we get

\[
\frac{p_c - p_0}{\alpha f_T} = \frac{Q}{2 \pi r_c T} \left( \log \left( \frac{R_i + r_i}{R_i + r_c} \right) - 1.07 \log \left( \frac{R_i}{R_i + r} \right) \right) + \frac{b Q^2}{4 \pi r^2 (R_i + r_c)} \left( \frac{1}{R_i + r_c} - \frac{1}{R_i + r} \right)
\]

[15]

Combining equations [4], [4], and [13] with [15], we obtain

\[
\frac{\alpha f_T}{\mu r_c} \left( \frac{p_c - p_0}{\mu r_c} \right) = 8000 \frac{r_i}{r_c} \left( \log \left( \frac{R_i + r_i}{R_i + r_c} \right) - 1.07 \log \left( \frac{R_i}{R_i + r} \right) \right) + 0.07 \frac{r_i}{(R_i + r_c) (R_i + r)}
\]

[16]
If we assume purely laminar flow in the entire flow region then the flow rate \( Q_{\text{in}} \) may be obtained from [5] by using the corresponding boundary conditions at the well and the counter of intake. Hence

\[
Q_{\text{in}} = \frac{2\pi k T}{\eta} \frac{(p_w - p_s)}{\log \left( \frac{R^2 - R_i^2}{R r_w} \right)} \tag{17}
\]

Therefore, from [13] and [17], we obtain the ratio

\[
\frac{Q}{Q_{\text{in}}} = \frac{8560}{R_i} \frac{r_i}{r_w} \left( \frac{\mu r_w}{g d^3 (p_c - p_w)} \right) \log \left( \frac{R^2 - R_i^2}{R r_w} \right) \tag{18}
\]

Introducing dimensionless quantity \( X \) and ratio \( Y \) such that

\[
X = \frac{g d^3 (p_c - p_w)}{\mu r_w} \tag{19a},
\]

\[
Y = \frac{Q}{Q_{\text{in}}} \tag{19b}
\]

and combining [18] with [16], we obtain an implicit relation

\[
1.07 X = \frac{X Y}{Z} \left[ \log \left( \frac{R_i}{r_w} + \frac{X Y}{8560 Z} \right) - \log \left( \frac{R_i}{r_w} + 1 \right) + 
\right.

-1.07 \log \left( \frac{X Y}{8560 Z} \right) +

+1.07 Z + 0.07 \frac{X Y}{8560 Z} \left( \frac{X Y}{8560 Z} - 1 \right) \right], \tag{20}
\]

where \( Z = \log \left( \frac{R^2 - R_i^2}{R r_w} \right) \).

It may be inferred from [19], that the value of \( X \) which is possible from physical considerations is \( X > 0 \), hence equation [20] becomes
1.07 Z = \left[Y + \frac{R_i}{r_w} \left( \frac{XY}{8560 Z} - \log \left( \frac{R_i}{r_w} + 1 \right) \right) \right] \\
- 1.07 \log \left( \frac{XY}{8560 Z} \right) + \\
+ 1.07 Z + \frac{0.07 \times XY}{8560 Z} \times \left( \frac{XY}{8560 Z} - 1 \right) \left( \frac{R_i}{r_w} + 1 \right) \times \frac{XY}{8560 Z} \\
\left[21\right]

5. - PARTICULAR CASE

If \( R_i = 0 \) that is, when the well is established concentrically with respect to the contour of intake, equation \left[21\right] reduces to

\[1.07 \log \left( \frac{R}{r_w} \right) = Y \left( 0.07 \frac{XY}{8560 \log \left( \frac{R}{r_w} \right)} \right) + \\
- 0.07 \log \left( \frac{XY}{8560 \log \left( \frac{R}{r_w} \right)} \right) + 1.07 \log \left( \frac{R}{r_w} \right) - 0.07 \left[22\right]

which corresponds to the non-linear laminar flow of fluid into a fully penetrating concentric well discussed by Upadhyay \cite{5}.

6. - DISCUSSION

From \left[19\right], it is evident that \( X \) depends on the density of the fluid, grain size of the medium, pressure difference of the system, viscosity of the fluid and well radius. Since \( \delta \) and \( \mu \) occur in higher powers in expression for \( X \), they highly affect the discharge. Moreover, from physical considerations it is obvious that \( X \) and \( Y \) are both positive.
Now, to get the definite idea of the result [21], we take \( \frac{E}{r_w} = 3 \cdot 10^7 \) that is the radius of contour of intake is 3000 times the radius of the well. Considering \( \frac{E}{r_w} = 10^2 \), the numerical values of \( Y \) are obtained corresponding to different values of \( X > 0 \) and have been graphically plotted in the form of curve – I (Fig. 2).

It is seen from curve I that as \( X \) increases, initially \( Y \) increases till it attains a maximum value 2.16 corresponding to \( X = 1.2691 \cdot 10^7 \), afterwards it decreases asymptotically. Thus in the former region \( 0 < X < 1.2691 \cdot 10^7 \), the discharge increases as \( X \) increases, that is, when the density of the fluid, grain size of the medium and pressure difference of the system increases, viscosity of the fluid and the well radius decreases. In the later region \( X > 1.2691 \cdot 10^7 \) the influence of non-linear laminar flow is reversed. Thus, it may be concluded that in case of non-linear laminar flow, the flow pattern is characterised by two different zones in which, discharge exhibits opposite character.
7. - COMPARISON

To examine as to how the position of the well affects the discharge into it, we consider the cases $\frac{R_1}{r_w} = 0$ and $\frac{R_1}{r_w} = 10$. These cases have been graphically represented by dotted curve and curve-II respectively in Fig. 2. Hence, it is inferred that as the well approaches the contour of intake the discharge increases abruptly as compared to that into a well concentrically established with respect to the contour of intake. From physical consideration, the result is quite obvious and consistent with that obtained by Polubarinova-Kochina in case of laminar flow.

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REFERENCES