ELECTROMAGNETIC FIELDS PRODUCED BY A CABLE (CARRYING AN ALTERNATING CURRENT) PLACED ON A GROUND WITH AN INTERSTRATUM

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FIRST PART

§ 1. The theory of the conduction of a. c. through the ground, by inductive influence of a cable infinitely long (parallel, or placed on the horizontal surface of a conducting ground) was the object of classical investigations.

Particularly, Haberlandt (¹) examined the case of an homogeneous ground of everywhere uniforme conductivity (σ_2), except near the surface, where it acts as covering, thin, conducting layer: σ_1, such that: $\sigma_1 \ll \sigma_o$.

The problem of determining the electromagnetic field in any point of the space, with varying of the physical geometric characteristics of this covering stratum, with rectilinear, long cable (displacement currents zero), carrying a. c.: $(R_e \cdot I e^{j\omega t})$, is reduced to a plane problem.

A more recent study of Evans (²) follows that of Haberland: both theories remove the limitation of considering a " limiting layer " (infinitely conductive and infinitely thin); this double limit is consequence of considered unfit boundary conditions, as it was remarked [2]. With other boundary conditions, it will be possible bear in mind, on the contrary, layers of " *finite thicknesses*: d ", placed on indefinite substrata of *different conductivity* (the limitation in [1] and [2]: $\sigma_2 \ll \sigma_1$, is here removed, with regard at least to asymptotic equations).

We shall examine, therefore, the cases of conductive or dry coverings, (with underlying conductive water creek, or mineral metallic sheet).

While in this first Part, we give solutions of the problem of the outcropping, in the second Part, we shall give that one of the problem of the " *embedded interstratum* " *with any resistivity contrast in respect to the embedding medium.*

We suppose to have a coordinate system (fig. 1), with origin at the ground-surface (x, z), below the rectilinear cable, which is taken at a distance h above the surface, and parallel to z axis, with y axis vertical to up.

The distances of a measurement-point $P(x, y)$, respectively from the cable and from its specular image in respect to (x, z), are designated by: r_0, r_1; the " numerical coordinates " by (ξ, η, ζ), assuming as " numerical unitarian length ": $\gamma = c\,(4\,\pi\,\omega\,\sigma_1)^{-1/2}$, (that gives the maximum

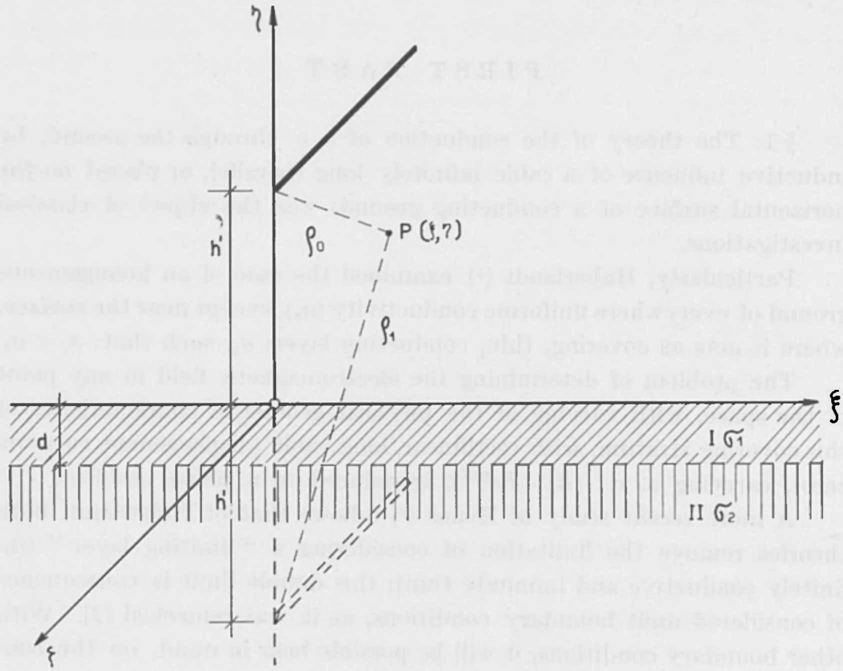

Fig. 1

depth for which the current penetrating in the ground still keeps a valuable magnitude). In a normal ground ($\varrho = 10^4\ \Omega$ cm.), with increasing frequencies from 50 to 150, 500, 1000 ($p \cdot \text{sec.}^{-1}$), γ reaches values of 500, 290, 160, 110 meters, while if ϱ increases, a little too, (e. g. $\varrho = 4.10^4\ \Omega$ cm., γ increases, in this case, the double. Our geometrical values will be reduced to " numerical values ", by means of:

$$x = \xi\,\gamma\ ,\ y = \eta\,\gamma\ ,\ r_0 = \varrho_0\,\gamma\ ,\ r_1 = \varrho_1\,\gamma\ ,\ h = h'\,\gamma\ ,\ d = d'\,\gamma\ .$$

The electric field (e. f.) E is reduced to the only component E_z (that we shall designate directly by E), which satisfies (see [2]) the fundamental equation:

$$\frac{\partial^2 E}{\partial \xi^2} + \frac{\partial^2 E}{\partial \eta^2} = j\,k\,E \quad ; \quad j = \sqrt{-1} \quad ; \qquad [0]$$

$$k = \begin{cases} 0 & \text{in air} \\ 1 & \text{in I} \\ \sigma_2/\sigma_1 & \text{in II} \end{cases} .$$

At the surface of separation (air-outcropping, layer-sublayer), the tangential components of e. f. $\overline{E}$, and the magnetic induction $\overline{B}$, and those normal of $\overline{B}$ are continuous.

The vectors $\overline{E}$, $\overline{B}$ are zero at ∞, and in the vicinity of the cable, B follows the Biot-Savart's law. It is convenient to introduce, in the place of $\overline{E}$, a proportional function: $c^{-1} \cdot 2\,\omega\,j\,\Phi$, (Gauss'units of measure).

Steady and boundary conditions will be expressed in terms of Φ, Φ_1, Φ_2, according to the region under consideration: air, media I, II.

The Φ, Φ_1, Φ_2 solutions are integral expressions, the arbitrary constants of which are determined with the boundary conditions.

The " numerical integrations " being laborious, Evans limits himself to an only typical example: valuation of the e. f. intensity $\overline{E}$, at the surface of the ground, below to cable, as function of the thickness " d " of the conducting outcropping layer, placed on substratum of very high resistivity. This in normal operative conditions and for normal ground:

$$\omega = 2\,\pi\,f = 6{,}28 \cdot 10^3 \text{ rad. sec}^{-1} \quad ; \quad \sigma_1 = 9 \cdot 10^8 \text{ u. e.s} \quad ;$$

$$h' = c^{-1}\,h\,(4\,\pi\,\omega\,\sigma_1)^{1/2} = 0{,}281 \quad ; \quad d' = c^{-1}\,d\,(4\,\pi\,\omega\,\sigma_1)^{1/2} = 2{,}81 \cdot 10^4 \cdot d\ .$$

Evans lists for E, aside from the factor $(4\,\omega\,I \cdot c^{-1})$, the following values:

d in cm	10	500	1000	1500	2000	∞
$\lvert E \rvert$	1,58	1,48	1,38	1,31	1,21	1,04 .

The electric field intensity E is approximately 50 percent greater at the surface of the thin conducting layer, 10 cm thick, than it is at the surface of the infinitely thick conductor, having the same conductivity.

The progressive increase of the thickness of conducting layer involves a diminution of the e. f. (measurable at the surface), of gradient zero or almost, from a certain " critical thickness " forward. This is an e. m. characteristic behaviour of the " thin layer ", of greater indicative moment than it is for the " thick layers ", in conformity with that we put recently in evidence with dipolar excitors: [5].

§ 2. We consider now the Haberlandt-Evans' integral expression of the e. f.: $\bar{E}$ (of difficult pratical use): [1].

If we put:

$$p = h' + \gamma \quad , \quad a = (j + \beta^2)^{1/2} \quad , \quad \theta = (\beta^2 + j\,k)^{1/2}$$

(β = real positive parameter), we obtain:

$$\left.\begin{aligned} &\Theta/I = \ln(\varrho_0/\varrho_1) + 2\int_0^\infty f(\beta)\,\exp.(-\beta\,p)\cos(\xi\,\beta)\,d\beta \\ &f(\beta) = \Big[(\theta - a)\exp.(-a\,d') - (\theta + a)\exp.(a\,d')\Big] \Big/ \Big[(\theta - \\ &\qquad - a)(a - \beta)\exp.(-a\,d' + (\theta + a)(a + \beta)\exp.(a\,d')\Big]\ . \end{aligned}\right\} \qquad [1]$$

It is well know, from the theory of the asymptotic expansions, that, for sufficiently great values of $(p^2 + \xi^2)^{1/2}$, i. e. so great that we may regard the $f(\beta)$ as a slowly variable function in comparison with the expression: exp. $(-\beta\,p)\cos(\xi\,\beta)$, a asymptotic expansion of the integral of [1], may be obtained, replacing the $f(\beta)$ by its power-series expansion around the origin.

In this manner, we obtain:

$$2\int_0^\infty f(\beta)\,\exp.(-\beta\,p)\cos\beta\,\xi\,d\beta \sim \sum_{n=0}^\infty f^{(n)}(o)\cdot\varphi_n \qquad [2]$$

with:

$$\begin{aligned} \varphi_n = \frac{2}{n\,!}\int_0^\infty \beta^n \exp.(-\beta\,p)\cos\beta\,\xi\,d\beta = \frac{1}{n\,!}\int_0^\infty \beta^n \exp.\Big[-(p + j\,\xi)\,\beta\Big]d\beta + \\ + \frac{1}{n\,!}\int_0^\infty \beta^n \exp.\Big[-(p - j\,\xi)\,\beta\Big]\,d\beta = (p + j\,\xi)^{-(n+1)} + (p - j\,\xi)^{-(n+1)}\ . \end{aligned} \qquad [2']$$

If only the first two terms of the series are taken, we have:

$$2 \int_0^\infty f(\beta)\, \text{exp.}\,(-\beta p) \cos \beta \xi \, d\beta \simeq 2\, p\, (p^2 + \xi^2)^{-1} \cdot f(o) + \qquad [2']$$

$$+ 2\,(p^2 - \xi^2)\,(p^2 + \xi^2)^{-2} \cdot f'(o) \; .$$

In order to calculate $f(o)$, $f'(o)$, it is convenient, to expand numerator and denominator as functions of β, considering only the terms of first order. Aside from terms β^2, we have for α and θ: $\alpha = \varepsilon$, $\theta = \varepsilon \sqrt{k}$, where we put: $\varepsilon = \sqrt{j} - (j+1)/\sqrt{2}$.

Hence, aside always from terms β^2:

$$f(\beta) = \frac{\left(\sqrt{k} - 1\right) \text{exp.}\,(-\varepsilon d') - \left(\sqrt{k} + 1\right) \text{exp.}\,(\varepsilon d')}{\varepsilon \left|\left(\sqrt{k} - 1\right) \text{exp.}\,(-\varepsilon d') + \left(\sqrt{k} + 1\right) \text{exp.}\,(\varepsilon d')\right| - \beta \left|\left(\sqrt{k} - 1\right) \text{exp.}\,(-\varepsilon d') - \left(\sqrt{k} + 1\right) \text{exp.}\,(\varepsilon d')\right|} \qquad [3]$$

If we let λ and U denote the expressions:

$$\lambda = \left(\sqrt{k} - 1\right) / \left(\sqrt{k} + 1\right) \quad ;$$
$$U = (1 - j) \left|\lambda - \text{exp.}\,(2\,\varepsilon\, d')\right| / \sqrt{2} \left|\lambda + \text{exp.}\,(2\,\varepsilon\, d')\right| \; , \qquad [4]$$

[3] becomes:

$$f(\beta) = U / (1 - \beta\, U) \; .$$

Hence, we may see that the necessary condition, in order to $f(\beta)$ may be regarded as a slowly variable function in comparison with: exp. $(-\beta p)$ cos $(\beta \xi)$, is the following:

$$(p^2 + \xi^2)^{1/2} \gg U \; , \qquad [5]$$

(that involves limitations of frequency).

Since, if (d') and (k) are not, contemporaneously, too little, $|U|$ is of the order of unit magnitude, condition [5] becomes:

$$(p^2 + \xi^2) \gg 1 \; , \qquad [5a]$$

while, if (d') and (k) are very little, [5] becomes:

$$\left(d' + \sqrt{k}\right) (p^2 + \xi^2)^{1/2} \ll 1 \; . \qquad [5b]$$

In this case, we have, for $|U|$, the order of magnitude: $(d' + \sqrt{k})^{-1}$, as it ean easily be shown from [4], if we put in the numerator: $\lambda \sim 1$, $\exp.(2\,\varepsilon\, d') \simeq 1$; in the denominator:

$$\lambda \sim -1 + 2\sqrt{k} \quad , \quad \exp.(2\,\varepsilon d') \sim 1 + 2\,\varepsilon\, d'$$

Having dehmited, in this manner, the "range of validity" of our asyptotic formula, we may write its expression. From [3′], we have immediately $f(o) = U$, $f'(o) = U^2$, thence we obtain, with regarding [2′] and [1]:

$$\Phi / I \simeq \ln(\varrho_0 / \varrho_1) + 2\,p\,U / (p^2 + \xi^2) + 2\,(p^2 - \xi^2)\,U^2 / (p^2 + \xi^2)^2 \,. \qquad [6]$$

The field, for $p = 0$ (cable placed on the earth, and also measurements made upon the soil), may be expressed by means of [5] as follows:

$$\left(\Phi / I\right)_{p=0} \simeq -2\,U^2\,\xi^{-2} \,. \qquad [7]$$

I. e.: *The electric field shows a decrement inversely proportional to the square of the distance from the inducing indefinite cable, placed on the ground surface.*

We could make use of this, in practice, in order to verify, among others, if the distances are sufficiently great to permit the application of [7]. With other words, it may be convenient to measure the e. f. in several points, at different distance, in order to obtain, eventually, with extrapolation, the value: $-2\,U^2 = \lim_{\xi \to \infty} (\Phi / I)$.

For infinite thickness, (uniform ground), [7] becomes:

$$\Phi_n / I \simeq 2\,j\,\xi^{-2} \,. \qquad [7']$$

If we let E_n denote the e. f., for $d = \infty$, from [7], [7′], we may obtain for the intensity $\overline{E}$: $(E / E_n = \Phi / \Phi_n)$:

$$E = V^2 E_n \;, \qquad [8]$$

$$V = -j\,(1 + j)\,U / \sqrt{2} = (\exp.\,2\,\varepsilon\,d' - \lambda) / (\exp.\,2\,\varepsilon\,d' + \lambda) \,.$$

It is particularly interesting to know its behaviour as function of "d" and "k".

Put $\delta = d' \sqrt{2}$, [8] gives easily:

$$| E / E_n | = V^2 = \tag{9}$$

$$= [\exp. 2\,\delta - 2\,\lambda \exp. \delta \cos \delta + \lambda^2] / [\exp. 2\,\delta + 2\,\lambda \exp. \delta \cos \delta + \lambda^2] .$$

hence the following limiting values:

for: $\delta = 0$: $| E / E_n | = (\lambda - 1)^2 / (\lambda + 1)^2 = k^{-1}$

for: $\delta = \infty$: $| E / E_n | = 1$.

Now in order to obtain the complete diagram of $| E / E_n |$ as function of the thickness, it is necessary to determine the zeros of the derivative of [9], with respect to d.

An elementary calculation gives:

$$\exp. 2\,\delta\, tn\, (\delta + \pi / 4) = \lambda^2 , \tag{10}$$

for determining these zeros.

Eq. [10] has infinite zeros of which we consider only the positive ones. These zeros, corresponding to extreme values (minima and maxima) of $| E / E_n |$, will be designated by: $\delta_0 , \delta_1 , \ldots \delta_k , \ldots$, in rising order. For δ_0, and other δ_k, we may write, approximately:

$$\delta_0 \simeq 3\,\pi / 4 + \lambda^2 \exp. (- 3\,\pi / 2) \qquad [11_0]$$

$$\delta_k \simeq (k + 3 / 4)\,\pi \quad ; \quad (k = 1, 2, \ldots) . \qquad [11_k]$$

The corresponding values of the ratio $| E / E_n |$ that we designate with $| E / E_n |_k$, will be:

$$| E / E_n |_0 \simeq 1 + 2 \sqrt{2}\,\lambda \exp. (- 3\,\pi / 4) + 4\,\lambda^2 \exp. (- 3\,\pi / 2)$$

$$[12]$$

$$| E / E_n |_1 \simeq 1 - 2 \sqrt{2}\,\lambda \exp. (- 7\,\pi / 4) \ ; \ | E / E_n |_k \simeq 1 \ ; \ (k \geqslant 2) .$$

With other terms, for $\delta > \delta_1$, the oscillations, around the unit, of $| E / E_n |$ are so low that $| E |$ may be regarded substantially as constant. These semi-oscillations, (of period: π) decrease with geometrical succession, of ratio: $\exp. (- \pi) = 0{,}043$.

The profile of $| E / E_n |$, given by [9], is immediately deducible, with the possibility (new in respect of the past theories) of distinguishing the following cases:

a) $k < 1$ (*outcropping layer, of greater conductivity than the under-layer*): the ratio $| E / E_n |$ decreases (from $\delta = 0$, for which initial value it becomes: k^{-1}), to $| E / E_n |$ for $\delta = \delta_0$; henceforth it tends to increase very slightly as far as $| E / E_n |$, for $\delta = \delta_1$, (value very near by unit), and finally it oscillates, around the unit, with period 2π (so that it may be regarded as constant).

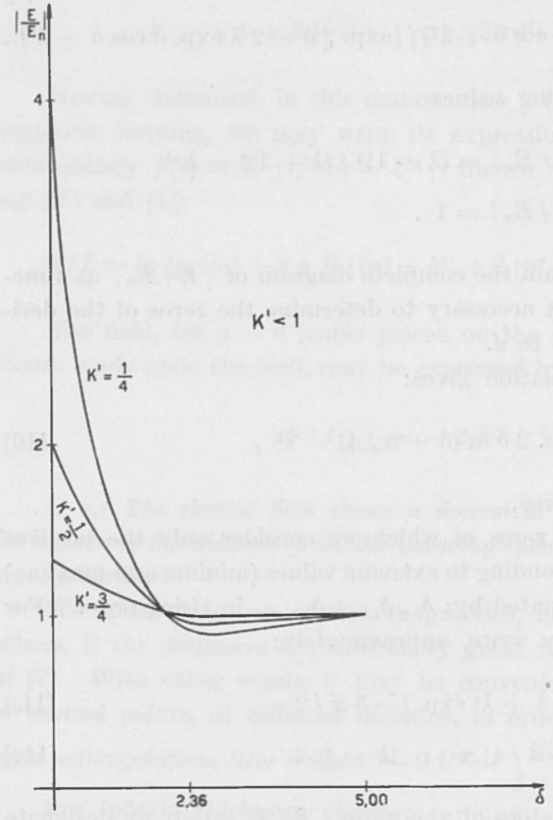

Fig. 2a

b) $k > 1$ (*insulating covering layer*): as in a), except the reciprocal exchange of the terms pertinent to "increase" and "decrease".

We give a table of $| E / E_n |$, calculated for a certain number of ratio k and of extreme values of: $\delta = 0$; δ_0; δ_1; ....

For these particular values of k, we have in practice:

$$\delta_0 = 3\pi / 4 = 2{,}36 \quad , \quad \delta_1 = 7\pi / 4 = 5{,}5 \; .$$

	$\mid E / E_n \mid$		
k	$\delta = 0$	$\delta = \delta_0$	$\delta = \delta_1$
1/4	4.000	0,915	1,004
1/2	2.000	0,955	1,002
3/4	1,333	0,981	1,001
3/2	0,667	1,027	0,999
2	0,500	1,047	0,998
4	0,250	1,093	0,996
∞	0	1,304	0,988

In conformity with the results deduced from [1], [2], [3], for $k < 1$, $| E / E_n |$ decreases with increasing of δ (thickness): see fig. 2.

Also for $k < 1$, Haberlandt too, in [1], comes to an e. m. potential which decreases, by the presence of a covering conducting layer, with increasing of its thickness, with concomitance of phase shifting.

If, on the contrary, the underlayer is of greater conductivity ($k > 1$) asymptotic $| E / E_n |$ increases with increasing of (δ) as far as a maximum, and then it decreases slightly: (see fig. 2).

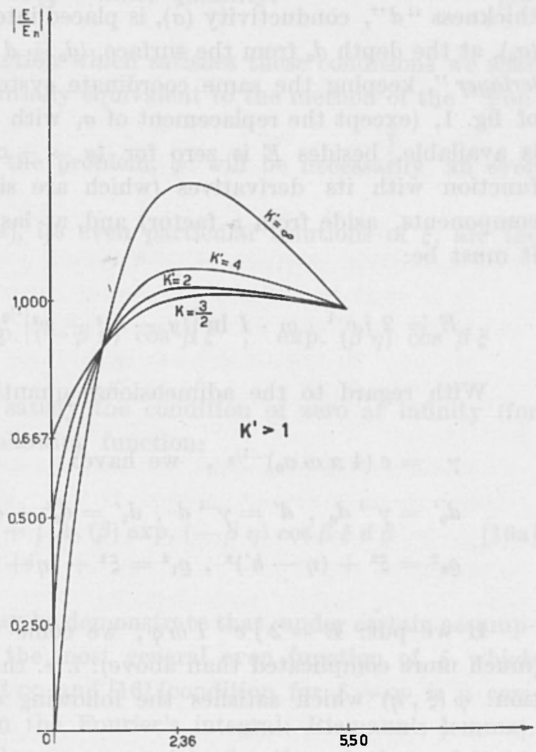

Fig. 2 b

However, the consideration of a superficial covering (with contrast of resistivity in respect of indefinite medium) is equivalent to a "field-correction", as Haberlandt made remark: [1]: ($c_0 = E_n - E$).

Consequently, the influence of a "conducting covering" (in respect of that one of considered homogeneous medium), as soon as its thickness increases, involves a progressive increase of c, from zero to the asymptote E_n; the influence, on the contrary, of a conducting substratum (e. g. water-layer) involves a profile of c, from E_n to a more or less accentuated minimum ($k = \infty, \rightarrow 1$).

Since the electric field (or induced voltage per lenght unit) is here simply the potential φ for unitarian current and frequence one: ($E = J \omega \varphi$), "*asymptotic potentials*" measurements (at the surface) will so permit electro-lithologic and dimensional identifications about the succession (in depth) of the first two ground layers.

SECOND PART

§ 3. We suppose now that the horizontal " outcropping layer " of thickness "*d*", conductivity (σ), is placed into the homogeneous ground, (σ_0), at the depth d_0 from the surface, ($d_0 + d - d_1$), *i. e. it becomes " interlayer "*, keeping the same coordinate system and the same symbols of fig. 1, (except the replacement of σ_1 with σ_0, and of σ_2 with σ) · (o) is available, besides $\overline{E}$ is zero for ($y \rightarrow \pm \infty$), and it is continuous function with its derivatives (which are simply the magnetic field components, aside from a factor) and, at last, near by ($x = 0$, $y = h$), it must be:

$$E = 2\,j\,c^{-1} \cdot \omega \cdot I \ln \left[(y - h)^2 + x^2\right]^{1/2} + \text{finite quantity.} \qquad [13]$$

With regard to the adimensional quantities, of unit:

$$\gamma = c\,(4\,\pi\,\omega\,\sigma_0)^{-1/2}\ , \quad \text{we have:}$$

$$d_0' = \gamma^{-1}\,d_0\ ,\ d' = \gamma^{-1}\,d\ ,\ d_1' = d_0' + d' = \gamma^{-1}\,d_1\ ,\ k = \sigma_1\,\sigma_0^{-1}$$

$$\varrho_0^2 = \xi^2 + (\eta - h')^2\ ,\ \varrho_1^2 = \xi^2 + (\eta + h')^2\ .$$

If we put: $E = 2\,j\,c^{-1}\,I\,\omega\,\varphi$, we come to formulate our problem (much more complicated than above): i. e. the determination of a function: $\varphi\,(\xi\,,\eta)$ which satisfies the following differential equations:

$$\vartheta^2\,\varphi\,/\,\vartheta\,\xi^2 + \vartheta^2\,\varphi\,/\,\vartheta\,\eta^2 = 0 \quad ,\ \text{for: } \eta > 0 \qquad [14a]$$

$$\vartheta^2\,\varphi\,/\,\vartheta\,\xi^2 + \vartheta^2\,\varphi\,/\,\vartheta\,\eta^2 = j\,\varphi \quad ,\ \text{for: } -d_0' < \eta < 0 \qquad [14b]$$

$$\vartheta^2\,\varphi\,/\,\vartheta\,\xi^2 + \vartheta^2\,\varphi\,/\,\vartheta\,\eta^2 = j\,k\,\varphi \quad ,\ \text{for: } -d_1' < \eta < -d_0' \qquad [14c]$$

$$\vartheta^2\,\varphi\,/\,\vartheta\,\xi^2 + \vartheta^2\,\varphi\,/\,\vartheta\,\eta^2 = j\,\varphi \quad ,\ \text{for: } \eta < d_1' \qquad [14d]$$

which are obtained from [1], after changement of the variables in [1], satisfying the following boundary conditions for φ:

$$\varphi \rightarrow 0 \ \text{ if } \ \xi \rightarrow \pm \infty \quad , \ \text{or} \ \ \eta \rightarrow \pm \infty\ . \qquad [15]$$

Let φ be continuous with its derivative: $\vartheta\,\varphi\,/\,\vartheta\,\eta$, (the continuity of the derivative: $\vartheta\,\varphi\,/\,\vartheta\,\xi$ results already as consequence of the continuity of φ) for: $\eta = 0$, $\eta = -d_0'$, $\eta = -d_1'$, and besides, in the vicinity of $\xi = 0$, $\eta = h'$, let be:

$$\varphi = \ln \varrho_0 + \text{finite quantity}. \qquad [16]$$

In order to find a function which satisfies these conditions we shall follow a proceeding substantially equivalent to the method of the " Fourier's Transformata ".

By the symmetry of the problem, φ will be necessarily an even function of ξ.

Referring now to [14a], its even particular solutions of ξ. are the following:

$$\ln \varrho_0 \quad , \quad \ln \varrho_1 \quad , \quad \exp. (-\beta\,\eta) \cos \beta\,\xi \quad , \quad \exp. (\beta\,\eta) \cos \beta\,\xi$$

the last of which do not satisfy the condition of zero at infinity (for $\eta \rightarrow \infty$), hence also the following function:

$$\varphi = \ln (\varrho_0 / \varrho_1) + \int_0^\infty A_0 (\beta) \exp. (-\beta\,\eta) \cos \beta\,\xi\, d\,\beta \qquad [16a]$$

will satisfy Eq. [14a]. It may be demonstrate that, under certain assumptions of regularity [16] is the most general even function of ξ which satisfies [14a], conditions at ∞ and [16] [condition for $\xi \rightarrow \infty$ is a consequence of a theorem on the Fourier's integral: Riemann's lemma).

In order to write analogous expressions for the solutions of [14b], [14c], [14d], we premise an observation on the roots of complex numbers.

By $(a + j\,b)^{1/2}$ will then be designated the principal root, i. e. the root having the real part positive. It is expressed by:

$$(a \pm j\,b)^{1/2} = \left[\frac{1}{2}\left(\sqrt{a^2 + b^2} + a\right)\right]^{1/2} \pm j \left[\frac{1}{2}\left(\sqrt{a^2 + b^2} - a\right)\right]^{+1/2}$$

where the (+) sign refers to the case: $b \geqslant 0$, the (—) sign to the case: $b \leqslant 0$.

We put: $\alpha = (\beta^2 + j)^{1/2}$, $\theta = (\beta^2 + j\,k)^{1/2}$, it may be show, with substitution, that functions: $\exp. (\alpha\,\eta) \cos \beta\,\xi$, $\exp. (-\alpha\,\eta) \cos \beta\,\xi$, are solutions of [14b], [14d], while: $\exp. (\theta\,\eta) \cos \beta\,\xi$, $\exp. (-\theta\,\eta) \cos \beta\,\xi$, are solutions of [14c].

Therefore, the following formulas too:

$$\varphi = \int_0^\infty A_1(\beta) \exp. (-\alpha\eta) \cos\beta\xi\, d\xi + \int_0^\infty B_1(\beta) \exp. (\alpha\eta) \cos\beta\xi\, d\xi \quad [16b]$$

$$\varphi = \int_0^\infty A_2(\beta) \exp. (-\theta\eta) \cos\beta\xi\, d\xi + \int_0^\infty B_2(\beta) \exp. (\theta\eta) \cos\beta\xi\, d\xi \quad [16c]$$

$$\varphi = \int_0^\infty B_3(\beta) \exp. (\alpha\eta) \cos\beta\xi\, d\xi \quad [16d]$$

are respectively solutions of [14b], [14c], and [14d].

It is clear that, in [16d], we did not use the: exp. $(-\alpha\eta) \cos\beta\xi$, which for $\eta \to -\infty$ do not tend to zero.

It may be demonstrated that these formulas are those most general satisfying the equations [14b], [14c] and [14d], the conditions at infinity, and they are even functions of ξ.

However, we may come to analogous solutions applying the stricter (though more laborious) method of the " Fourier's Trasformata ".

In order to determine the functions: $A_i(\beta)$ and $B_i(\beta)$ there remains to apply the conditions of continuity: on the ground of this last, the following expressions must coincide:

1) for $\eta = 0$, the φ and $\frac{\partial\varphi}{\partial\eta}$, calculated from [16a], [16b];

2) for $\eta = -d_0'$, the φ and $\frac{\partial\varphi}{\partial\eta}$, calculated from [16b], [16c];

3) for $\eta = -d_1'$, the φ and $\frac{\partial\varphi}{\partial\eta}$, calculated from [16c], [16d].

On the whole, it's the question of 6 conditions for determining the 6 functions:

1) $\eta = 0)$ $\quad : \int_0^\infty A_0(\beta) \cos\beta\xi\, d\beta = \int_0^\infty A_1(\beta) \cos\beta\xi\, d\beta + \int_0^\infty B_1(\beta) \cos\beta\xi\, d\beta.$

$$-2h' / (h'^2 + \xi^2) - \int_0^\infty A_0(\beta)\beta \cos\beta\xi\, d\beta = -$$

$$-\int_0^\infty A_1(\beta)\,\alpha \cos\beta\xi\, d\beta + \int_0^\infty B_1(\beta)\,\alpha \cos\beta\xi\, d\beta$$

$$2)\ \eta = -d_0' : \int_0^\infty A_1(\beta) \exp.\ (\alpha d_0') \cos \beta\xi\, d\beta + \int_0^\infty B_1(\beta) \exp.\ (-$$

$$-\alpha d_0') \cos \beta\xi\, d\beta = \int_0^\infty A_2(\beta) \exp.\ (\theta d_0') \cos \beta\xi\, d\beta +$$

$$+ \int_0^\infty B_2(\beta) \exp.\ (-\theta d_0') \cos \beta\xi\, d\beta.$$

$$-\int_0^\infty A_1(\beta)\, \alpha \exp.\ (\alpha d_0') \cos \beta\xi\, d\beta + \int_0^\infty B_1(\beta)\, \alpha \exp.\ (-$$

$$-\alpha d_0') \cos \beta\xi\, d\beta = -\int_0^\infty A_2(\beta)\, \theta \exp.\ (\theta d_0') \cos \beta\xi\, d\beta +$$

$$+ \int_0^\infty B_2(\beta)\, \theta \exp.\ (-\theta d_0') \cos \beta\xi\, d\beta.$$

$$3)\ \eta = -d_1' : \int_0^\infty A_2(\beta) \exp.\ (\theta d_1') \cos \beta\xi\, d\beta + \int_0^\infty B_2(\beta) \exp.\ (-$$

$$-\theta d_1') \cos \beta\xi\, d\beta = \int_0^\infty B_3(\beta) \exp.\ (-\alpha d_1') \cos \beta\xi\, d\beta\,.$$

$$-\int_0^\infty A_2(\beta)\, \theta \exp.\ (\theta d_1') \cos \beta\xi\, d\beta + \int_0^\infty B_2(\beta)\, \theta \exp.\ (-$$

$$-\theta d_1') \cos \beta\xi\, d\beta = \int_0^\infty B_3(\beta)\, \alpha \exp.\ (-\alpha d_1') \cos \beta\xi\, d\beta\,.$$

From these relations among the integrals, we may easily come to the relations between A_i and B_i, remarking that all integrals have the form: $\int_0^\infty f(\beta) \cos \beta\xi\, d\beta$. Now, from theory of Fourier's integrals, we have: $\int_0^\infty f(\beta) \cos \beta\xi\, d\beta = 0$, for any $\xi > 0$, which involves: $f(\beta) = 0$, hence the equality among integrals involves the equality among the coefficients which multiply $(\cos \beta\xi)$.

It is still necessary to write only the term: $-2h' / (h'^2 + \xi^2)$ in form of $\int_0^\infty f(\beta) \cos \beta\xi\, d\beta$, which is soon made on the ground of:

$$2 \int_0^\infty \exp.\ (-h'\beta) \cos \beta\xi\, d\beta = \int_0^\infty \exp.\ \left|-h' - j\xi)\beta\right| d\beta +$$

$$+ \int_0^\infty \exp.\ \left|-(h' + j\xi)\beta\right| d\beta = 2h' / (h'^2 + \xi^2)\ .$$

We obtain, in order to determine A and B, the following system of linear equation:

$$A_0 = A_1 + B_1 \ ,$$

$$-2 \exp. (-h' \beta) - \beta A_0 = a (-A_1 + B_1) \ ,$$

$$A_1 \exp. (a d_0') + B_1 \exp. (-a d_0') = A_2 \exp. (\theta d_0') + B_2 \exp. (-\theta d_0') \ ,$$

$$a \left[-A_1 \exp. (a d_0') + B_1 \exp. (-a d_0')\right] = \theta \left[-A_2 \exp. (\theta d_0') + + B_2 \exp. (-\theta d_0')\right] \ , \qquad [17]$$

$$A_2 \exp. (\theta d_1') + B_2 \exp. (-\theta d_1') = B_3 \exp. (-a d_1') \ ,$$

$$\theta \left|-A_2 \exp. (\theta d_1') + B_2 \exp. (-\theta d_1')\right| = a B_3 \exp. (-a d_1') \ .$$

Since what more interesting is the field in air, we will limit ourselves to the determinations of A_0 from the above system.

Multiplying the penultimate of [17] by a, and subtracting from this the last, we obtain:

$$(\theta + a) A_2 \exp. (\theta d_1') - (\theta - a) B_2 \exp. (-\theta d_1') = 0 \ ,$$

$$B_2 = A_2 \exp. (2 \theta d_1') \cdot \{ (\theta + a) / (\theta - a) \} \ .$$

Substitution of last value in the third and the fourth of [17], gives for these:

$$A_1 \exp. (a d_0') + B_1 \exp. (-a d_0') =$$
$$= A_2 \left[(\theta + a) (\theta - a)^{-1} \exp. \{ (2 d' - d_0') \theta \} + \exp. (\theta d_0') \right] \ ,$$

$$a \left[-A_1 \exp. (a d_0') + B_1 \exp. (-a d_0')\right] =$$
$$= \theta A_2 \left[(\theta + a) (\theta - a)^{-1} \exp. \{ (2 d' - d_0') \theta \} - \exp. (\theta d_0') \right] \ .$$

Elimination of A_2 between these two equations, gives, after simplification (recollecting that: $d_1' - d_0' = d'$):

$$\left[\nu^{-1} \exp. (\theta d_1') - \nu \exp. (-\theta d_1')\right] A_1 \exp. (a d_0') +$$
$$+ \left[\exp. (\theta d') - \exp. (-\theta d')\right] B_1 \exp. (-a d_0') = 0 \ , \qquad [18]$$
$$\nu = (\theta - a) / (\theta + a) \ .$$

However, if the first two of [17] are solved for A_1, B_1, the results are the following:

$$A_1 = A_0 (\alpha + \beta) / 2 \alpha + \alpha^{-1} \exp. (- \beta h')$$
$$B_1 = A_0 (\alpha - \beta) / 2 \alpha - \alpha^{-1} \exp. (- \beta h') \ .$$

Substitution of these values in [18], and solution for A_0, gives finally: $A_0 = 2 f(\beta) \exp. (- \beta h')$

$$f(\beta) = \frac{- \left| \nu^{-1} \exp. (\theta d') - \nu \exp. (- \theta d') \right| \exp. (\alpha d_0') +}{(\alpha + \beta) \left| \nu^{-1} \exp. (\theta d') - \nu \exp. (- d') \right| \exp. (\alpha d_0') +} \qquad [19]$$
$$\frac{+ [\exp. \left| \theta d') - \exp. (- \theta d') \right| \exp. (- \alpha d_0')}{+ (\alpha - \beta) \left| \exp. (\theta d') - \exp. (- \theta d') \right| \exp. (- \alpha d_0')}$$

hence, after substitution in [14a], we have definitively for *the electric field in air*:

$$\varphi = \ln (\varrho_0 / \varrho_1) + 2 \int_0^\infty f(\beta) \exp. \left| - (h' + \eta) \beta \right| \cos \xi \beta \, d\beta \ . \qquad [20]$$

This formula coincides, for $d' \to + \infty$, with [30] of Evans, aside a factor (I), since our φ is the Φ / I in [2]. It may also be verified that, for $k = 1$, it coincides with *the Carson's formula for the normal field.*

It can be shown, besides, that for $\sigma \to \infty$, $(k \to \infty)$, $f(\beta)$ becomes independent of d' (thickness of interlayer), which is physically evident, since a perfectly conducting layer do not trasmit any field, (indepently of its thickness). Indeed, for $k = + \infty$, we have: $\nu = 1$ and:

$$f(\beta) = \left[- \exp. (\alpha d_0') + \exp. (- \alpha d_0') \right] / \left[(\alpha + \beta) \exp. (\alpha d_0') - \right.$$
$$\left. - (\alpha - \beta) \exp. (- \alpha d_0') \right]$$

according perfectly to the analogous formula, which, for $k \to \infty$ is obtained in [2].

§ 4. Now, let us examine (analogously to the first Part), the electric field at great distances from the cable.

As shown above in [2'], we have, for high values of: $\left| \xi^2 + (\eta + h')^2 \right|^{1/2}$:

$$\int_0^\infty f(\beta) \exp. \left[- \beta (\eta + h') \right] \cos \beta \xi \, d\beta \sim$$
$$\sim f(o) \cdot (\eta + h') / \left[\xi^2 + (\eta + h')^2 \right] + f^1(o) \left[(\eta + h')^2 - \xi^2 \right] / \left[(\eta + h')^2 + \xi^2 \right]^2 .$$

Substituting [21] in [20] and regarding only the case of cable placed on the ground ($h' = 0$) and of field-measurements on the ground ($\eta = 0$), we have:

$$\varphi = -2 f'(o) / \xi^2 , \qquad [22]$$

i. e. *the asymptotic field, here too, shows a decrement inversely proportional to the square of the distance.*

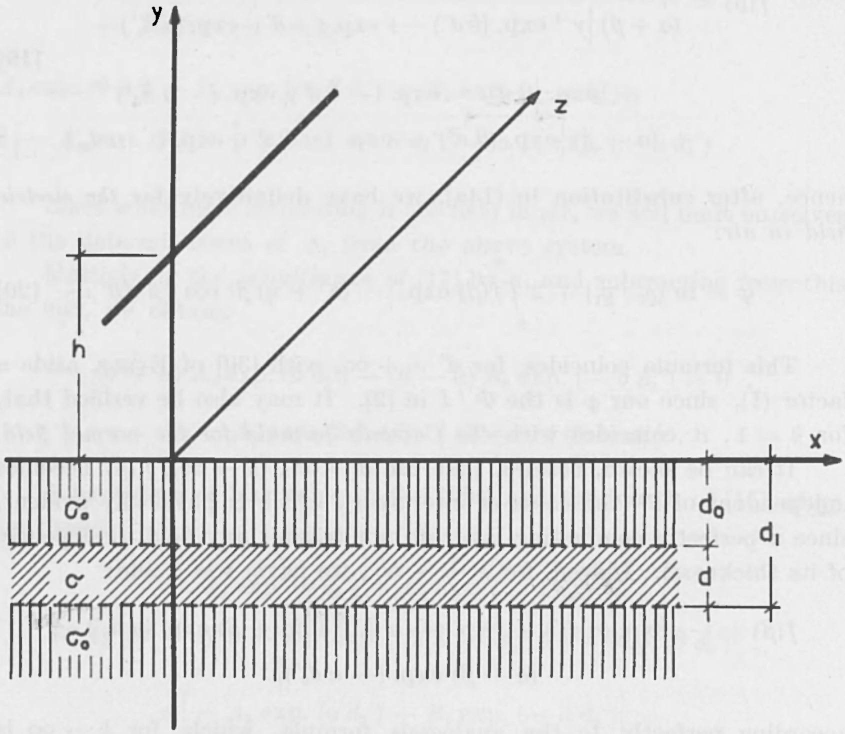

Fig. 3

As shown above in first Part, this proprierty may be used to verify for what distance the asymptotic formula is available.

However, the experimental value which should be compared with $f'(0)$, is simply the limit of $(-\varphi \xi^2 / 2)$, when $\xi \to \infty$, which may be obtained by extrapolation of the measurements of $(-\varphi \xi^2 / 2)$, graphically or with some other proceeding.

In order to express $f'(0)$ by means of d_0', d', k, it is convenient to introduce the function:

$$g(\beta) = \frac{-\left[\nu^{-1} \exp.(\theta d') - \nu \exp.(-\theta d')\right] \exp.(\alpha d_0') + \alpha\left[\nu^{-1} \exp.(\theta d') - \nu \exp.(-\theta d')\right] \exp.(\alpha d_0') + }{+\left[\exp.(\theta d') - \exp.(-\theta d')\right] \exp.(-\alpha d_0') + \alpha\left[\exp.(\theta d') - \exp.(-\theta d')\right] \exp.(-\alpha d_0')} \qquad [23]$$

and, in this case, $f(\beta)$ may be expressed by $g(\beta)$ as follows:

$$f(\beta) = g(\beta) / \left[1 - \beta g(\beta)\right] \quad ; \quad f'(\beta) = \left[g^1(\beta) + g'(\beta)\right] / \left[1 - \beta \cdot g(\beta)\right]^2$$
$$f'(0) = g'(0) + g^2(0) \ .$$

But: $g'(0) = 0$, since $g(\beta)$ is an even function of β; hence: $f'(0) = g^2(0)$; the [22] becomes: [22'] $\varphi \simeq -2\, g^2(0) / \xi^2$ where $g(0)$ must be calculated.

If we put: $\varepsilon = j^{1/2} = (1 + j) / \sqrt{2}$; $\lambda = (\sqrt{k} - 1) / (\sqrt{k} + 1)$; $\Delta = d' \sqrt{k} = c^{-1} d\, (4\pi\sigma\omega)^{1/2}$ (it must be observed that Δ depends upon σ and d, in the same manner as d_0' upon σ_0 and d_0), we have, for $\beta = 0$, $\alpha = \varepsilon$, $\theta = \varepsilon \sqrt{k}$, $\nu = \lambda$, $\theta d' = \Delta$, $g(0) = V\,(j - 1) / \sqrt{2}$,

$$V = \frac{\left[\lambda^{-1} \exp.(\varepsilon\Delta) - \lambda \exp.(-\varepsilon\Delta)\right] \exp.(\varepsilon d_0') - \left[\lambda^{-1} \exp.(\varepsilon\Delta) - \lambda \exp.(-\varepsilon\Delta)\right] \exp.(\varepsilon d_0') +}{-\left[\exp.(\varepsilon\Delta) - \exp.(-\varepsilon\Delta)\right] \exp.(-\varepsilon d_0') + \left[\exp.(\varepsilon\Delta) - \exp.(-\varepsilon\Delta)\right] \exp.(-\varepsilon d_0')} \qquad [23]$$

and [22'] becomes):

$$\varphi \simeq 2\, j\, V^2 / \xi^2 \ . \qquad [22'']$$

If the normal field ($k = 1$) is designated by φ_n, which involves: $\lambda = 0$, $V = 1$ the equation takes the form:

$$\varphi_n \simeq 2\, j / \xi^2 \ , \qquad [22''']$$

which coincides with [7'].

Now, bearing in mind that: $\varphi / \varphi_n = E / E_n$ (E_n = normal electric strenght), [22''] may be written as follows too:

$$E / E_n = V^2 \ , \qquad [24]$$

to which it is convenient to give an other easier analitical form.

We designate by F the complex expression (which however may be obtained from the measurements):

$$F = \left[1 - (E / E_n)^{1/2}\right] / \left[1 + (E / E_n)^{1/2}\right] \qquad [25]$$

$(E / E_n)^{1/2}$ = principal value.

From (24), we have:

$$F = (1 - V) / (1 + V) ,$$

or: [26]

$$F = \exp.(-2\varepsilon d_0') \left[\exp.(\varepsilon \Delta) - \exp.(-\varepsilon \Delta)\right] / \left[\lambda^{-1} \exp.(\varepsilon \Delta) - \lambda \exp.(-\varepsilon \Delta)\right] .$$

If we put:

$$\delta_0 = d' \sqrt{2} \; , \; \delta = \Delta / \sqrt{2} \; , \; \tau = - \ln |\lambda| = \ln \left| \frac{\sqrt{k} + 1}{\sqrt{k} - 1} \right| ,$$

[26] becomes:

$$F = \pm \left\{ Sh \left[\delta (1 + j)\right] \exp. \left[-\delta_0 (1 + j)\right] / Sh \left[\delta + \tau + j \delta\right] \right\} \qquad [27]$$

where the (+) sign refers to case $k > 1$ (*conducting layer*), that (—) one to case $k < 1$ (*insulating layer*). F consists in two factors, one of which depends only upon δ_0, the other only upon δ and k, through: τ.

It is necessary to find the expressions of modulus $|F|$ and the phase: (arg F). Since:

$$|Sh(a + jb)| = \left|Sh(a + jb)\, Sh(a - jb)\right|^{1/2} = \left|(Ch\, 2a - \cos 2b) / 2\right|^{1/2} \qquad [28]$$

$$\arg Sh(a + jb) = \arg \left[Sh(a \cos \beta) + j\, Ch(a \operatorname{sen} \beta)\right] = \operatorname{arctn} (tn \beta / th\, a)$$

from [27] we have:

$$|F| = \exp.(-\delta_0) \left\{ \left[Ch(2\delta) - \cos(2\delta)\right] / \left[Ch\, 2(\delta + \tau) - \cos 2\delta\right] \right\}^{1/2}$$

$$\arg F = -\delta_0 + \operatorname{arctn} \left[tn \delta / th \delta\right] - \operatorname{arctn} \left[tn \delta / th(\delta + \tau)\right]$$

[29]

$$\arg F = \begin{cases} \pi , & \text{if } k < 1 \\ 0 , & \text{if } k > 1 . \end{cases}$$

From the first of [29], it is seen that $|F| < 1$, and decreases very rapidly with increasing of δ_0 or τ; the increase of τ signifies the approximation of k to unit. Unfortunately, in the value of $|F|$, the cases $k > 1$, $k < 1$ are not distinguishable: indeed, approximating τ to zero (remaining δ_0 and δ costant), $|F|$ tendes to (exp. $(-\delta_0)$; since $\tau \to 0$, whether for $k \to 0$, or for $k \to \infty$, this signifies that a not little value of $|F|$ may denote the presence of highly conducting or highly insulating interlayer.

Therefore the phase of F, where the term "discontinuous": (π for $k < 1$, or 0 for $k > 1$), distinguishes clearly the two cases, is much more significative.

Besides, this phase depends in a very simple manner upon the depth δ_0 of interlayer.

Tough from [29] it is not possible to determine: δ_0, δ, τ in practice *the second expression of* [29] *will permit, by itself, to ascertain if the layer is conducting or insulating.*

In fact, in order to (F) may be measurable (considering $\sqrt{E} / \sqrt{E_n}$), neither τ, nor δ_0 may reach very high values.

Consequence of this is that also (arg. F) will remain low enough, hence (when $k > 1$, or $k < 1$), (arg. F) will be more near 0 or π, respectively.

The expression, independent of δ_0, which for $k < 1$ is always $> \pi$, could help:

$$\arg F - \ln|F| = \operatorname{arctn}\left[tn\delta / th\delta\right] - \operatorname{arctn}\left[tn\delta / th(\delta + r)\right] +$$

$$+ \frac{1}{2} \ln\left[Ch\, 2(\tau + \delta) - \cos 2\delta) / \left[Ch\,(2\delta) - \cos 2\delta\right]\right. , \qquad [30]$$

$$\arg F - \ln|F| = +\pi \text{ , if } k < 1 : = 0 \text{ , if } k > 1 .$$

Hence: $0 < \arg F - \ln|F| < \pi$ designates a "conducting layer". We give the formulas for two limiting cases:

a) δ very great, then [26] becomes:

$$F = \lambda \exp.\left[-\delta_0(1 + j)\right] \text{ , hence: } |F| = |\lambda| \exp.(-\delta_0) \text{ ;}$$

$$\arg F = -\delta_0 = +\pi \text{ if } k < 1 \text{ , } = 0 \text{ if } k > 1 .$$

b) δ, τ little, then we may put $(Sh\,\delta = \delta)$ and [27] becomes:

$$F = \pm\,\exp.\left[-\,\delta\,(1+j)\right]\delta\,(1+j)\,/\left[\delta\,(1+j)+\tau\right]$$

$$|F| = \delta\,\exp.(\delta_0)\,/\left[\delta^2+\tau\,\delta+\tau^2\,/\,2\right]^{1/2}\ ;$$

$$\arg F = -\,\delta_0 + \operatorname{arctn}\left[\tau\,/\,(\tau+2\,\delta)\right] = \begin{cases}+\pi\\ 0\end{cases}.$$

If $k > 1$, and δ, τ are little, we have a thin conducting interlayer of high conductivity. If $k < 1$, and δ, τ are little, we have a interlayer of very low conductivity in respect of that of the embedding ground, but not necessarily very thin.

Indeed, δ is proportional to the conductivity of the interlayer, and hence it may be very little, without that " d " becomes necessarily little.

Our problem is completely resolved.

SUMMARY

The A. studies the e. m. field, generated by a rectilinear infinitely long cable (carrying an alternating current), placed on the surface of a ground with horizontal infinitely long " interlayer ". The interlayer can be " surface outcropping " or be placed into the homogeneous ground, at a determinate depth.

After elimination of usual limitations about the minimum thickness of this layer (infinitely little and infinitely conducting), the A. solves the problem pertinent to the difference of conductivity ($\gtrless 0$) with the underlayer. In the first Part, the A. comes to the solution of the " integral problem " of outcropping layer, or of covering layer of an homogeneous isotropic ground.

In the second Part, the solution of the " integral problem " of the interstratum placed at determinate depth, of finite thickness, with any difference of conductivity in respect to the embedding medium, is given.

In § 1), the A., at first, follows the Haberlandt's and Evans's theories, departing from a fundamental deduction by Evans about the e. f., as function of the thickness of the superficial conducting layer.

This justifies introduction of the physical-geometric notions, as " thin layer ", " critical thickness ", binded to characteristic e. m. properties.

The A. attained recently, with some variants, the same deductions,

using " dipolar exciters ". See: " The function of the thin conducting layer " (with effect of e. m. pseudo-resonance, low frequency) — Geofisica Pura ed Applicata — Vol. 31 — Milano (1955).

In § 2), after simple asymptotic formulas (regarding e. f. E), the A studies the e. f. E as function of the thickness of the layer and of the underlayer, giving particular diagrams.

Not perfect coincidence with Evans's results, may be easily justified, but it is well to put in evidence that, substantially, the same law of the variation of E with thickness of superficial layer is obtained. This law is here extended to the case (also important) of the superficial layer with greater resistivity than the underlayer.

In § 3), the A. develops the problem of real " interstratum ", placed at determinate depth, in a indefinite homogeneous ground. Following same analitical proceding of the first Part, the expression of e. f. in air is given, which coincides with that Evans's one, for infinite thickness of the interlayer, and with that Carsons's one of normal e. f.

In § 4), the asymptotic e. f. (i. e. at great distance from the cable) is examined, finding analogous formulas to the first Part.

Finally, the A. establishes the modalities for determining depth and electric characteristic of the interlayer (with reference to those of embedding medium), giving, the first time, the complete solution of the problem.

RIASSUNTO

L'A. studia il campo e. m. (elettromagnetico) provocato da un cavo infinitamente lungo (percorso da corrente alternata a bassa frequenza), posto sulla superficie orizzontale d'un terreno con interstrato infinitamente esteso parallelo alla superficie del suolo.

L'interstrato di spessore finito può affiorare alla superficie o considerarsi posto nel suolo omogeneo ad una profondità data, situazione per la prima volta qui esaminata.

Dopo eliminate le limitazioni abituali sullo spessore minimo di questo strato (in generale assunto dai precedenti AA. infinitamente piccolo e infinitamente conduttivo), l'A. risolve il problema generale dell'interstrato a spessore finito a conduttività differenziale pure finita, affiorante o no.

Nella 1ª Parte si da la soluzione quando l'interstrato affiora in superficie, nella 2ª Parte si da la soluzione completa quando l'interstrato è profondo.

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