GENERAL SOLUTIONS OF SECOND ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

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INTRODUCTION

Let us consider the second order linear homogeneous differential equation

$$\frac{d^2 V}{dt^2} + P_1(t)\frac{dV}{dt} + Q_1(t)V(t) = 0 \qquad [01]$$

and exclude the trivial solution $V = o$; with the assumption

$$V(t) = U(t) exp\left(-\frac{1}{2}\int P_1\, dt\right) \qquad [02]$$

equation (01) takes the form which is called *normal* (*), namely:

$$\frac{d^2 U}{dt^2} + P(t)U = 0 \qquad [03]$$

The coefficient $P(t)$ is referred to as the *invariant* of the differential equation, and it is easily found its expression, that is:

$$P(t) = Q_1(t) - \frac{1}{4}P_1^2(t) - \frac{1}{2}\frac{dP_1(t)}{dt} \qquad [04]$$

The [03] transformation is possible under the scarcely limiting condition that $P_1(t)$ is derivable.

It is rather idle to remember that a very large number of practical problems of physics are represented by equations which can be reduced to an (homogeneous or less) « normal » type, like (0,3); as an example of uttermost importance, see any factorizable case of the Schrödinger equation.

(*) Cf. *E. Kamke*, Differ. Gleich., Leipzig 1944 Bd I, 119; and *S. Brodetsky*, Proc. Edimb. Math. Soc. 34 (1916) 45.

1° *Series expansion for the solutions of second order linear differential equation* $U'' = [k^2 + f(t)]\, U(t)$.

Let

$$\frac{d^2 V}{d t^2} = (k^2 + f(t))\, V \qquad [1]$$

be a « normal » differential equation where $f(t)$ is a known function, k a const., and assume

$$U = e^{kt}\, \Phi \ ; \quad k = \pm\sqrt{k^2} \qquad [2]$$

defining the function

$$\Phi = exp \int \gamma\, dt \qquad [3]$$

in such a way that when $f \equiv$ const., then $\Phi \equiv 1$

Particularly, if $f \equiv O$, equation [1] reenters in the common case with harmonic solutions e^{kt}.

With the process outlined above by substitution of [2] into [1], one may obtain the *Riccati's* equation (*):

$$\gamma' + \gamma^2 + 2\, k\, \gamma = f \qquad [4]$$

Suppose now to determine such a value t_k of the variable t that $f_k = f\,(t_k)$ obeys to the condition:

$$f_k = \gamma^2(t_k) = \gamma_k^2 \ ; \qquad [5]$$

in this case the solution of equation [4] at the point t_k will be:

$$\gamma_k' + 2\, k\, \gamma_k = 0 \qquad [6]$$

that is

$$\gamma_k = exp\,(-2kt_k - 2a) \ ; \quad 2\,a = const. \qquad [7]$$

and therefore

$$f\,(t_k) = exp\,(-4\, t_k - 4a) \qquad [8]$$

from which we may deduce t_k if the arbitrary constant a is fixed.

The arbitrariness of a assures us about the exsistence of the point t_k. It is then possible an expansion of γ into a Taylor's series in the vicinity of t_k:

$$\gamma = \gamma_k + \frac{(t - t_k)}{1\,!}\, \gamma'_k + \frac{(t - t_k)^2}{2\,!}\, \gamma_k'' + \ldots \qquad [9]$$

(*) By the way, the present method may be useful for solving Riccati's first order equations.

where

$$\gamma_k = e^{-2kt_k - 2a} = A$$

$$\gamma'_k = -2kA \qquad [10]$$

$$\gamma_k'' = f_k' - 2k\gamma_k' - 2(\gamma\gamma')_k = \qquad [11]$$

$$= f_k' + (2k)^2 A + 2(2k)A^2 \qquad [12]$$

$$\gamma_k''' = f_k'' - 2k\gamma_k'' - (\gamma^2)_k'' =$$

$$= f_k'' - (2k)f_k' - (2k)^3 A - 2(2k)^2 A^2 - (\gamma^2)_k'' . \qquad [13]$$

Observe now that

$$(\gamma^2)' = 2\gamma\gamma'$$

$$(\gamma^2)'' = 2\gamma\gamma'' + 2\gamma'^2$$

$$(\gamma^2)''' = 2\gamma\gamma''' + 6\gamma'\gamma'' \qquad [14]$$

$$(\gamma^2)^{\mathrm{IV}} = 2\gamma\gamma^{\mathrm{IV}} + 8\gamma'\gamma''' + 6\gamma''^2$$

which, for $\gamma_k \rightarrow 0$, $\gamma_k' \rightarrow 0$, entraines $(\gamma^2)^{(n)} \longrightarrow 0$, so that we may write symbolically

$$(\gamma^2)^{(n)}_k = (\gamma_k + \gamma_k)^{(n)} = \sum_{o}^{m}{}_r \binom{n}{r} \gamma_k^{(n-r)} \gamma_k^{(r)} \qquad [15]$$

Thus we obtain:

$$\gamma_k^{(n)} = f_k^{(n-1)} - 2k\gamma_k^{(n-1)} - (\gamma^2)_k^{(n-1)} = \qquad [16]$$

$$= f_k^{(n-1)} - 2k\gamma_k^{(n-1)} - \sum_{o}^{n-1}{}_r \binom{n-1}{r} \gamma_k^{(n-1-r)} \gamma_k^{(r)}$$

For the determination of A it is necessary to recall the condition $\Phi=1$ when $f =$ const. If we consider that [2] and [3] give $U = exp\,(kt + \int \gamma\, dt)$, and that this expression must coincide with e^{kl} when $f = const = 0$, then we recognize that the value $\gamma_k = 0$ verifies our assumption, id est

$$A = 0 \quad ; \quad a = +\infty \qquad [17]$$

Thus we get:

$$\gamma_k = 0 \qquad [18]$$

$$\gamma_k' = 0 \qquad [19]$$

$$\gamma_k'' = f_k' \qquad [20]$$

$$\gamma_k''' = f''_k - 2kf'_k \qquad [21]$$

The condition

$$\gamma_k = 0 \qquad [22]$$

signifies that t_k is a zero of the function $f(t)$ and therefore it requires the reliability of [23]:

$$f_k = 0 \qquad [23]$$

id est $f(t)$ *must have at least one zero*. If this is not the case, instead of the equation [1] we may deal with

$$\frac{d^2 U}{d t^2} = [k^2 + h^2 + (f - h^2)]\, U \qquad [24]$$

where $h^2 = const$ is the minimum distance of $f(t)$ from zero, or it is the mean value of $f(t)$ or even an absolute value of $f(t)$ higher than the minimum (absolute) value of $f(t)$, or other equivalent definition. It may be shown that the apparent arbitrariness of const. h^2 or even of $K^2 = k^2 + h^2$ is removed by the border conditions of the actual problem which leads to the equation [1], (see § 5), and to the eigenvalues of the differential equation.

With the notations

$$F(t) = f(t) - h^2 \quad ; \quad K^2 = k^2 + h^2 \qquad [25]$$

i. e. adding and subtracting h^2 in the bracket of the second member of [1], we obtain the equation [26] identical to the given equation:

$$\frac{d^2 V}{d t^2} = \left(K^2 + F(t)\right) V \qquad [26]$$

which satisfies howerer the desired condition [23] (*). In the following text we always write $f(t)$, which must be understood as changed into $F(t)$ when $f(t)$ has no zero.

Now let us punt:

$$\gamma^{(n)} = f^{(n-1)} - p f^{(n-2)} + p^2 f^{(n-3)} - \ldots\ldots + (-1)^{n-1} p^{n-1} f + \qquad [27]$$
$$+ (-1)^n (p^n \gamma + p^{n-1} G - p^{n-2} G' + \ldots) - G^{(n-1)}$$

where

$$G'_k = 2 \gamma_k \gamma'_k = 0 \qquad [28]$$

$$p = 2k \qquad [29]$$

(*) It is obvious the at the points t_k one must find $F(t_k) = o$ when $f(t)$ has no zero.

Because of the recurrent formulae [15] and of the conditions $\gamma_k = 0$, $\gamma'_k = 0$, all derivatives $G^{(n)}$ vanish, so that, omitting for the sake of simplicity all indexes k, we obtain:

$$\left\{\begin{aligned} &\gamma = 0 \\ &\gamma' = 0 \\ &\gamma'' = f' \\ &\gamma''' = f'' - p f' \\ &\gamma^{IV} = f''' - p f'' + p^2 f' \end{aligned}\right. \qquad [30]$$

and therefore

$$\gamma^{(n)} = \sum_{0}^{n-1}{}_{r} (-1)^r p^r f^{(n-1-r)} \qquad [31]$$

whence, because of [9], indicating with t_o a particular value of t for which $f(t)$ or $F(t)$ vanish,

$$\gamma = \sum_{1}^{\infty}{}_{n} \frac{(t-t)^n}{n!} \sum_{0}^{n-1}{}_{r} (-1)^r (2k)^r f_o^{(n-1-r)} \qquad [32]$$

As a final result from [2] [3] [31] [32] we have:

$$U = exp \left\{ k t + \sum_{1}^{\infty}{}_{n} \sum_{0}^{n-1}{}_{r} \frac{(-1)^r}{n!} (2k)^r f_o^{(n-1-r)} \int (t-t_o)^n dt \right\} = \qquad [33]$$

$$= exp \left\{ k t + \sum_{1}^{\infty}{}_{n} \sum_{0}^{n-1}{}_{r} \frac{(-1)^r}{(n+1)!} (2k)^r f_o^{(n-1-r)} (t-t_o)^{n+1} \right\} =$$

$$= exp \} k t + S \{$$

with $S = \log \Phi$·

The formula [33] is the desired expansion. Actually two expansions are possible in correspondence with the two signs of $k = \pm \sqrt{k^2}$. It is well-known that this double value leads to the determination of the two arbitrary constants connected to a second order differential equation.

It is not unuseful to remark, either from an algebraic or from a practical point of view for the calculation of the solutions, that as matter of fact our problem can just be considered as resolved with the formula [33]. Further developments are only possible semplifications or explicit expressions of this formula.

It is worth noting that the solution [33] is particularly suitable for numerical calculations, as it depends only upon kt and upon the derivatives $f_0^{(n)}$. As far as the derivation is usually an operation of much easier calculation than the integration, the usefulness of [33] seems to be evident.

2° *Explicit expression of the solution* [33]

The expression appearing in [33]

$$\log \Phi = S = \sum_{1}^{\infty}{}_{n} \sum_{0}^{n-1}{}_{r} (-1)^{r} \frac{p^{r}}{(n+1)!} (t - t_0)^{n+1} f_0^{(n-1-r)} \qquad [34]$$

for successive r values, recalling that $f_0^{(0)} = f_0 = 0$, with the notation

$$\tau = t - t_0 \qquad [35]$$

gives the successive summations, which may be disposed along a triangle:

$$S = \sum_{2}^{\infty}{}_{n} \frac{\tau^{n+1}}{(n+1)!} f_0^{(n-1)} - p \sum_{3}^{\infty}{}_{n} \frac{\tau^{n+1}}{(n+1)!} f_0^{(n-2)} + \qquad [36]$$

$$+ p^2 \sum_{4}^{\infty}{}_{n} \frac{\tau^{n+1}}{(n+1)!} f_0^{(n-3)} - \ldots.$$

Now we may substitute for n, through the individual sums, another symbol in order to reduce to an unique value, for instance $(n + 1)$, all the $(n - r)$ symbolic exponents of f_0' that is

$$S = \sum_{0}^{\infty}{}_{n} \frac{\tau^{n+3}}{(n+3)!} f_0^{(n+1)} - p \sum_{0}^{\infty}{}_{n} \frac{\tau^{n+4}}{(n+4)!} f_0^{(n+1)} + \qquad [37]$$

$$+ p^2 \sum_{0}^{\infty}{}_{n} \frac{\tau^{n+5}}{(n+5)!} f_0^{(n+1)} - \ldots$$

By adding in column the coefficients of the same derivative $f_0^{(n+1)}$ and because of one of the wellknown theorems for the double series, we get:

$$S = \sum_{0}^{\infty}{}_{n} \sum_{0}^{\infty}{}_{r} \frac{\tau^{n+3+r} (-1)^{r} p^{r}}{(n+3+r)!} f_0^{(n+1)} \qquad [38]$$

If now we put

$$q = p \tau . \qquad [39]$$

and

$$\tau^{n+3} B_3 (n,q) = \sum_{0}^{\infty}{}_{r} \frac{(-1)^{r} p^{r} \tau^{n+3+r}}{(n+3+r)!} = \tau^{n+3} \sum_{0}^{\infty}{}_{r} \frac{(-1^{r} q^{r}}{(n+3+r)!} \qquad [40]$$

that is

$$B_3(n,q)=\sum_{r=0}^{\infty}\frac{(-1)^r q^r}{(n+3+r)!}\;{}^{(\star)}$$

then [41]

$$S=\sum_{n=0}^{\infty}\tau^{n+3}B_3(n,q)f_o^{(n+1)}$$

But

$$B_3(n,q)=\sum_{r=0}^{\infty}\frac{(-1)^r q^r}{(n+3+r)!}=\frac{1}{q^{n+3}}\sum_{r=0}^{\infty}\frac{(-1)^r q^{n+3+r}}{(n+3+r)!}=$$

$$=\frac{(-1)^{n+3}}{q^{n+3}}\left[\sum_{r=0}^{\infty}-\sum_{r=0}^{2}\right]\frac{(-1)^r q^r}{r!}=\frac{(-1)^{n+3}}{q^{n+3}}\left(e^{-q}-\sum_{r=0}^{n+2}\frac{(-1)^r q^r}{r!}\right) \qquad [42]$$

so that

$$B_3(n,q)=\frac{(-1)^{n+3}}{q^{n+3}}\left(e^{-q}-\sum_{r=0}^{n+2}\frac{(-1)^r q^r}{r!}\right)\left(\substack{\star\\ \star}\right) \qquad [43]$$

Therefore the expression S, and consequently U, is completely determined if the values of $f_o^{(n+1)}$ are known. Formula [43] means also that for $n\to\infty$ the $B_3(n,q)'s$ tend towards $(-1)^{n+1}(e^{-q}-e^{-q})/q^{n+3}\to 0/q^{n+3}$.

3° B$_3$ (n, q) *reduction to the incomplete* Γ *function.*

With a methode indicated by *Cauchy and Saalschütz* (see Whittaker & Watson - Modern Analysis, Cambridge, 1950 pg. 244) if we write

$$\beta(n,q)=(-1)^{n+3}q^{n+3}B_3(n,q) \qquad [44]$$

we obtain

$$\Gamma(z)=\lim_{q\to\infty}\int_0^q q^{z-1}\beta(n,q)\,dq \qquad [45]$$

(*) It is generally defined:

$$B_s(n,q)=\sum_{r=0}^{\infty}\frac{(-1)^r q^r}{(n+s+r)!}=B_n(s,q), \quad [41\ A]$$

A particular value is

$$B_o(o,q)=e^{-q},.[41\ B]$$

(⁑) The sum in the brackets may be formally definide as the incomplete exponential function, since its value tends towards e^{-q} for $n\to\infty$.

where z is real and greater than 1, and n is the integer *immediately* superior to z if z is not an integer. If z is an integer number, then, obviously,

$$n = +z > 1 \qquad [46]$$

Now integrating by parts we obtain:

$$\int_o^q q^{z-1}\beta(n,q)\,d\,q = \left[\frac{q^2}{z}\beta(n,q)\right]_o^q + \frac{1}{z}\int_o^q q^2\beta(n-1,q)\,d\,q = \qquad [47]$$

$$= \frac{1}{z}q^z\beta(n,q) + \frac{1}{z}\int_o^q q^z\beta(n-1,q)\,d\,q$$

Consider the *case* $q \geqslant o$.

The first member of [47] is indistinguishable from the definition of the incomplete Γ function (or « digamma ») which is indicated by $\Gamma\ (q, z)$ so that:

$$\Gamma(q,z) = \frac{1}{z}q^z\beta(n,q) + \frac{1}{z}\Gamma(q,z+1) \qquad [48]$$

and therefore when

$$q \geqslant o$$

we get

$$\beta(n,q) = z\,q^{-z}\Gamma(q,z) - q^{-z}\Gamma(q,z+1) = q^{-n}[n\,\Gamma(q,n) - \Gamma(q,n+1)] \quad [49]$$

The $\Gamma\ (q, n)$ are tabulated by *K. Pearson*, Tables of the incomplete Γ - functions, « Biometrika », Cambridge, 1951, who gives the functions

$$I(u,n) = \frac{\Gamma(u,n)}{\Gamma(u,\infty)} \qquad [50]$$

where

$$\Gamma(u,\infty) = \Gamma(u) = (u-1)! \qquad [51]$$

and

$$u = \frac{q}{\sqrt{n+1}} \qquad [52]$$

It is always possible to use the formula [49] even for $q < o$, though for negative values of q it is necessary to calculate $I\ (-u, n)$

as function of $I\,(u, n)$ through Pearson's tables, with the formula (Pearson op. cit. pg. VI, formula [III] ter):

$$\frac{I(-u,p)}{(-u)^{p+1}} = \frac{[I'(o,p)]^2}{I'(u,p)} \cdot \left\{ 1 + \frac{u^2(p+1)^2}{(p+2)^2(p+3)} + \frac{u^4(p+1)^3}{(p+2)(p+3)^2(p+4)(p+5)} + \right.$$

$$\left. + \frac{u^6(p+1)^4}{(p+2)(p+3)(p+4)^2(p+5)(p+6)(p+7)} + \ldots \right\} \qquad [53]$$

where

$$I'(u,p) = \frac{I(u,p}{u^{p+1}} \qquad [54]$$

The functions $I'\,(u, p)$ may be found in table III of the just quoted Pearson's book; the particular value $I'\,(o, p)$ is given by the formula

$$I'(o,p) = (p+1)^{\sqrt{p-1}} / \Gamma(p+1) \qquad [55]$$

4° *A remarkable reduction.*

The function defined in [3], say

$$\Phi(\tau) = exp \sum_{o}^{\infty}{}_{n}\, \beta(n,q) f_o^{(n+1)} \tau^{n+3} \; ; \quad U = e^{kt}\, \Phi(\tau) \qquad [56]$$

with

$$\beta(n,q) = \sum_{o}^{\infty}{}_{r} \frac{(-1)^r q^r}{(n+3+r)!} \qquad [57]$$

may be, at its turn, developed in MacLaurin series:

$$\Phi(\tau) = 1 + \sum_{1}^{\infty}{}_{s} \frac{\tau^s}{s!}\, \Phi_o^{(s)} \qquad [58]$$

As $\Phi_o = \Phi(o) = 1$, and as the derivatives are

$$\frac{d}{d\tau} \sum_{o}^{\infty}{}_{n}\, \beta(n,q) f_o^{(n+1)} \tau^{n+3} =$$

$$= \sum_{o}^{\infty}{}_{n} \sum_{o}^{\infty}{}_{r} \frac{(-1)^r p^r \tau^{n+2+r}}{(n+2+r!)} f_o^{(n+1)}$$

and generally

$$\frac{d^s}{d\tau^s} \Phi(\tau) = \sum_{o}^{\infty}{}_{n,r} (-1)^r p^r f_o^{(n+1)} \frac{\tau^{n+2+r-s}(n+3+r)(n+2+r)\ldots(n+3+r-s)}{(n+3+r)!} =$$

$$= \sum_{o}^{\infty}{}^{n,r} (-1)^r p^r f_o^{(n+1)} \frac{\tau^{n+3+r-s}}{(n+3+r-s)!} \qquad [59]$$

so it is immediately seen that at the point $\tau = 0$, all addends for which $n + 3 + r - s = 0$, do vanish. The only one which does not vanish, has, as exponent for r,

$$n + 3 + r - s = 0 \qquad [60]$$

and reduces to

$$(-1)^r p^r f_o^{(n+1)} = (-1)^{s-n-3} p^{s-n-3} f_o^{(n+1)}$$

if, we assume, as well as it is generally assumed in any series expansion, (for instance in the cos x series for $x \to o$):

$$\frac{o^o}{o!} = 1.$$

Therefore

$$\frac{d^s \Phi(\tau)}{d\tau^s} = \sum_{o}^{\infty}{}_n (-1)^{s-n-3} p^{s-n-3} f_o^{(n+1)} \qquad [61]$$

Now on one hand the exponent r cannot be less than zero, therefore $r = s - n - 3 \geqslant 0$; on the other hand $r = 0$, if $n = s - 3$, but n cannot be less than zero, so that

$$n \geqslant s - 3 \geqslant 0$$

and therefore *all derivatives with* s < 3 *vanish.*

By putting

$$s = 3 + u \qquad (u = 0, 1, \ldots)$$

we get the condition

$$n + r = u = 0, 1, 2, \ldots$$

and then for $s \geqslant 3$ we obtain

$$\left(\frac{d^{3+u} \Phi}{d\tau^{u+3}} \right)_o = \sum_{o}^{u}{}_n \sum_{o}^{u-n>o}{}_u (-1)^{u-n} p^{u-n} f_o^{(n+1)} \qquad [62]$$

with the condition, for $(u - n)$, to be always positive or zero.

Therefore:

$$\begin{aligned} &\Phi'_o = 0 \\ &\Phi_o'' = 0 \\ &\Phi_o''' = f_o^{(1)} \qquad\qquad [63] \\ &\Phi_o^{IV} = -p f_o^{(1)} + f_o^{(2)} \\ &\Phi^V = p^2 f_o^{(1)} - p f_o^{(2)} + f_o^{(3)} \end{aligned}$$

and generally, for $n \geqslant 0$

$$\Phi_o^{(3+u)} = \overset{u}{\underset{0}{\Sigma}}_r \, (-1)^{u+r} \, p^{u-r} f^{(r+1)} \qquad [64]$$

so that our MacLaurin's series becomes

$$\Phi(\tau) = 1 + \overset{\infty}{\underset{0}{\Sigma}}_s \frac{\tau^{3+s}}{(3+s)!} \overset{s}{\underset{0}{\Sigma}}_r \, (-1)^{s+r} \, p^{s-r} f_o^{(r+1)} \qquad [65]$$

and defining the *polynomials*

$$\gamma_s(p, f_o) = \overset{s}{\underset{0}{\Sigma}}_r \, (-1)^{s+r} \, p^{s-r} f_o^{(r+1)} \qquad [66]$$

we finally obtain the most practical form of our resolving formula, say:

$$\Phi(\tau) = 1 + \overset{\infty}{\underset{0}{\Sigma}}_s \frac{\tau^{3+s}}{(3+s)!} \, \gamma_s(p, f_o) \qquad [67]$$

In the case of p *imaginary*, we may solve the polynomials $\gamma_s(p, f_o)$ in a real part $R_s(p, f_o)$ and in an imaginary part $I_s(p, f_o)$.

Actually, if *s is even*:

$$\begin{aligned} \gamma_s(i\,p, f_o) &= \overset{s}{\underset{0}{\Sigma}}_l \, (-1)^l \, i^l p^l f_o^{(s+1-l)} = \qquad [68] \\ &= f_o^{(s+1)} - i\,p\,f_o^{(s)} - p^2 f_o^{(s-1)} + i\,p^3 f_o^{(s-2)} + p^4 f_o^{(s-3)} \ldots = \\ &= f_o^{(s+1)} - p^2 f_o^{(s-1)} + p^4 f_o^{(s-3)} \ldots + i\,\{p\,f_o^{(s)} - p^3 f_o^{(s-2)} + p^5 f_o^{(s-4)}\} = \\ &= \overset{\frac{1}{2}s}{\underset{0}{\Sigma}}_l (-1)^l p^{2l} f_o^{(s+1-2l)} - i \overset{\frac{1}{2}(s-2)}{\underset{0}{\Sigma}}_l \, (-1)^l p^{2l+1} f_o^{(s-2l)} \; ; \end{aligned}$$

id est

$$\begin{aligned} \gamma_{2n}(i\,p, f_o) &= \overset{n}{\underset{0}{\Sigma}}_l \, (-1)^l p^{2l} f_o^{(2n+1-2l)} - i \overset{n-1}{\underset{0}{\Sigma}}_l \, (-1)^l p^{2l+1} f_o^{(2n-2l)} = \\ &= R_{2n}(p, f_o) - i\,I_{2n}(p, f_o) \; ; \end{aligned} \qquad [69]$$

if *s is odd*, $s = 2n + 1$

$$\begin{aligned} \gamma_{2n+1}(i\,p, f_o) &= \overset{2n+1}{\underset{0}{\Sigma}}_l \, (-1)^l \, i^l p^l f_o^{(2n+2-l)} = \qquad [70] \\ &= \overset{n}{\underset{0}{\Sigma}}_l \, (-1)^l p^{2l} f_o^{(2n+2-2l)} - i \overset{n}{\underset{0}{\Sigma}}_l \, (-1)^l p^{2l+1} f_o^{(2n+1-2l)} = \\ &= R_{2n+1}(p, f_o) - i\,I_{2n+1}(p, f_o) \end{aligned}$$

where

$$\left\{\begin{aligned} R_{2n}(p,f_o) &= \sum_{o}^{n}{}_{l}\,(-1)^l p^{2l} f_o^{(2n+1-2l)} \\ I_{2n}(p,f_o) &= \sum_{o}^{n}{}_{l}\,(-1)^l p^{2l+1} f_o^{(2n-2l)} \end{aligned}\right.$$

$$\left\{\begin{aligned} R_{2n+1}(p,f_o) &= \sum_{o}^{n}{}_{l}\,(-1)^l p^{2l} f_o^{(2n+1-2l)} \\ I_{2n+1}(p,f_o) &= \sum_{o}^{n}{}_{i}\,(-1)^l p^{2l+1} f_o^{(2n+1-2l)} \end{aligned}\right. \qquad [71]$$

and as a *general result* we obtain

$$\Phi(\tau) = 1 + \sum_{o}^{\infty}{}_{s} \frac{\tau^{3+s}}{(3+s)!}(R_s - i\,I_s) = \qquad [72]$$

$$= 1 + \sum_{o}^{\infty}{}_{s} \frac{\tau^{3+2s}}{(3+2s)!} R_{2s}(p,f_o) - i \sum_{o}^{\infty}{}_{s} \frac{\tau^{4+2s}}{(4+2s)!} I_{2s+1}(p,f_o)\,.$$

5° *Determination of* h². — *Eigenvalues.*

Since we have assumed that f_k equals zero at the particular point $t = t_k$, it follows that at this point the function Φ takes obviously the value

$$\Phi_k = 1\,. \qquad [73]$$

Now, when the function U, which enters into the normal differential equation [1], has two given values as border conditions, or even when it is given a value of U and of its derivative at a known point, then these two given values, determine the general solution of the differential equation.

Namely, the two signs of const. $\sqrt{k^2}$ determine two solutions either of then corrisponding to one of the two signs of k.

Indicating by U_1 and U_2 these two solutions, the general solution will be:

$$U = MU_1 + NU_2 \qquad [74]$$

where M, N are constants. Because of the condition [73] we get now, for instance,

$$\left\{\begin{aligned} & M\,e^{kt}{}_k + N\,e^{-kt}{}_k = U_k = U(k) \\ & M\,{}^{ka}\,\Phi_a + N\,e^{-ka}\,\Phi_a{}^{\star} = U_a = U(a) \\ & M\,e^{kb}\,\Phi_b + N\,e^{-kb}\,\Phi_b{}^{\star} = U_b = U(b) \end{aligned}\right. \qquad [75]$$

in the case of two given values of U at two points like a and b. Here $\Phi_a^\star$ [or $\Phi_b^\star$] is the value of $\Phi\ (a)$ [or $\Phi\ (b)$] with $-\,k$ instead of $+\,k$.

In a quite similar way we may get a similar system like [75] for the case in which the known values are $U\ (a)$ and $U'(a)$.

The system [75] consists of three equations with four unknowns M, N, K, U_k; thus one of these is indetermined. However this indeterminacy may be removed by the fact that U is always determined by an arbitrary factor apart, so that instead of the system [75] one may deal with the system (for example)

$$\left\{\begin{array}{l} e^{kt_k}+p\ e^{-kt_k}=U_k \\ e^{ka}\ \Phi_a+p\ e^{-ka}\ \Phi_a^\star=U_a \\ e^{kb}\ \Phi_b+p\ e^{-kb}\ \Phi_b^\star=U_b \end{array}\right. \qquad [76]$$

which has three unknowns only, namely p, U_k, k. Of course [76], can be replaced by a symmetrical system in which p is the coefficient of the factor e^{+kt}, whilst e^{-kt} has the coefficient 1.

Consequently we may assume the solution

$$U=e^{kt}\ \Phi+p\ e^{-kt}\ \Phi^\star \qquad [77]$$

a factor apart.

It is well-known that when the const. k deduced from [76] has a discrete or even a continuous set of values, these are referred to as the *eigenvalues* of the given equation.

By the way, it is worth while to observe, too, that te indeterminacy of the factor of U is usually removed with the well-known *condition of normalization* of the function U, accordingly to the actual problem represented by the given differential equation.

6° *Multiple sets of eigenvalues*

In the former treatement we have simply assumed the exsistence of such a point t_k as to yield the *caracteristic equation*

$$f(t)=0\,. \qquad [78]$$

Now, let us consider the more general case, when the caracteristic equation [78] has more than one (real or complex) solution. Then there are as so many systems like [75] as how many values t_k are determined by the caracteristic equation [78].

If system [75] leads to a set of eigenvalues, then there are as so many sets of eigenvalues as how many are the t_k; that is, one gets a *multiple set of eingenvalues.*

Denote by K_α the single elements of one set, corresponding to a single $t_\alpha = t_{k_\alpha}$ solution of the caracteristic equation [78].

If $f(t)$ is a polynomial, then the $t_\alpha{}'_s$ are the roots of the algebraic equation $f(t) = 0$. Therefore there are some known relations between the roots $t_\alpha{}'_s$ and the coefficients of the polynomial $f(t)$.

It is quite obvious that even these relations lead to corresponding relations between the eigenvalues of the different sets.

Then we may say that the sets are *linked* together.

Otherwise, when $f(t)$ is not a polynomial and no relation is reliable between the roots of $f(t) = 0$, then the sets of eigenvalues will be called *unlinked.*

Roma, December 1954.

SUMMARY

Here man searches the solutions of the 2nd order differential equations of the type $U'' + PU' + QU = 0$ *where* P, Q *are known function of the independent variable; that type is of uttermost importance for physics.*

The solutions are found as a special series of powers whose coefficients can be evaluated by the values of P *and* Q *and the derivatives of then colculated in some determined points. Formulae* (33) *and* (72) *of the text are reliable for numerical calculations. Some properties of the eigenvalues are also found.*