

Rayleigh wave dispersion equation with real terms in a layered transversely isotropic half space

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ABSTRACT

We consider transversely isotropic media with vertical axis of symmetry (VTI). Solutions of P-SV equation of motion in a homogeneous VTI medium contain depth decay factors r_1 and r_2 , which sometimes become complex depending on medium parameters. In this case, r_1 and r_2 are complex conjugates. Using this property, we obtain dispersion equation of Rayleigh waves with real terms for a layered VTI half space through Thomson-Haskell method with reduced delta matrix. Phase and group velocities as well as surface ellipticity of Rayleigh waves are computed in real domain for a few oceanic and continental VTI structures of the earth. Present computation in real domain is similar to that in a layered isotropic half space using the same method. Thus it is presumed that such computation in a layered VTI half space will allow efficient evaluation of a VTI structure of the earth.

1. Introduction

Transverse isotropy with vertical axis of symmetry (VTI) or radial anisotropy of the crust and upper mantle has been noted through the incompatibility to fit Rayleigh and Love wave dispersion with an isotropic structure [Anderson 1961]. During last few decades this incompatibility shows the variation of VTI in the upper mantle of various regions [Kustowski et al. 2008] indicating mantle flow and discriminating different geodynamic processes and thermo-chemical convective models [Auer et al. 2015, Khan et al. 2011]. VTI in the crust has also been seen in several regions [Shapiro et al. 2004, Raykova and Nikolova 2003, Moschetti et al. 2010]. In exploration seismology, better seismic images using reflection data are obtained with VTI in the subsurface [Thomsen 1986, Alkalifah and Tsvankin 1995].

The estimation of underground structure using surface waves is based on an inversion in which observed surface wave dispersion is compared with theoretical one for an assumed structure usually with a

horizontally layered medium. In an isotropic layered half space, computation of theoretical surface wave dispersion by Thomson-Haskell matrix method or T-H method [Haskell 1953] has proven to be a powerful tool. The restatement of recursive relation in terms of delta matrix (also called compound matrix or second order sub-determinant) is a significant improvement in T-H method [Dunkin 1965, Thrower 1965]. This was further improved by reduced delta matrix [Watson 1970]. Stability of T-H method at high frequencies can be improved by reduction of deeper layers, whose contributions to dispersion are negligible [Bhattacharya 1987]. The reflection-transmission method in a layered isotropic half space is a complex arithmetic scheme proposed by Kennett [1974] which is stable at very high frequencies. At present it has been developed as a generalized reflection-transmission coefficient method, or, in brief RT method [Kennett 1983, Luco and Aspel 1983, Chen 1993]. However, this method is least efficient of all improved methods [Buchen and Ben-Hardor 1997]; Pei et al. [2008] improved this method to increase the speed of computation. Efficiency of this method is low because of complex arithmetic and dispersion equations consist of real and imaginary parts, both of which are to be considered during root finding.

As indicated in the next section, solutions of P-SV equations in a homogeneous VTI layer contain $\exp(\pm kr_j z)$, where z is the vertical coordinate, k is wave number and $j = 1, 2$; r_j may be considered as depth-decay factor. Rayleigh wave dispersion equation in a layered VTI half space becomes complex when r_j becomes complex in one or more layers [Anderson 1961, 1965]. In such a layered half space, Anderson [1961] and Harkrider and Anderson [1962] computed surface wave dispersion in a layered VTI half space by extending the T-H method. Harkrider and Anderson [1962] replaced complex r_j in each layer by one from

equivalent isotropic layer to avoid the computation in a complex domain. The method of Anderson [1961] has been improved by doubled T-H method [Ke et al. 2011] and also by delta-matrix method [Ikeda and Matsuoka 2013]. In both the studies, since r_j becomes complex in some layers, complex numbers are used in the computations; as in RT method, roots need to be evaluated both from real and complex parts of the dispersion equation. For Love wave dispersion computation using T-H method in a layered VTI half space, there is no such problem and dispersion equation is obtained with real terms [Anderson 1962]. However, such dispersion computation for Rayleigh waves is being hampered due to complex terms in the dispersion equation. On the other hand, the T-H method for Rayleigh waves and its improvements [Buchen and Ben-Hardor 1997] in an isotropic medium are so popular that that it has been used even to evaluate radially anisotropic medium, where it is assumed that vertically and horizontally polarised S-wave velocities (β_V and β_H) are different but vertical and horizontal P-wave velocities (α_V and α_H) are equal; it is also assumed that the fifth parameter $\eta = 1$ [Shapiro et al. 2004, Raykova and Nilolova 2003, Moschetti et al. 2010, Xie et al. 2013]. Xie et al. [2013] mentioned that this assumption is physically unrealistic in real assemblages, where S-wave anisotropy would be accompanied by P-wave anisotropy with η differing from unity [Babuska and Cara 1991]. Further, Bhattacharya [2015] has shown that dispersion curves with such assumption for VTI structure are close to the correct result for Love waves; but such assumption is not satisfactory for Rayleigh waves. It has also been observed that this assumption overestimates S-wave anisotropy.

When r_j is complex in a VTI medium, r_1 and r_2 are complex conjugates. Using this property, we get a layer matrix as well as a reduced delta layer matrix, where each element is real even if r_j is complex in any layer. In a layered VTI half space, we obtain a Rayleigh wave dispersion equation which remains real with or without presence of complex r_j in any layer. Dispersion curves as well as surface ellipticity are computed in real domain for oceanic and continental layered VTI half space.

2. Basic equations

Let us consider a homogeneous VTI medium with anisotropic parameters as α_H , β_V , ξ , ϕ and η [Takeuchi and Saito 1972, Nishimura and Forsyth 1989]. Solutions of such medium were obtained by Anderson [1961], who used elastic stiffness coefficients c_{mn} as medium parameters [Love 1927]. These two representations of elastic

parameters are related by $\alpha_H^2 [=c_{11}/\rho]$, $\alpha_V^2 [=c_{33}/\rho]$, $\beta_V^2 [=c_{44}/\rho]$, $\beta_H^2 [=c_{11}-c_{12}]/2\rho$ and $\eta [=c_{13}/(c_{11}-2c_{44})]$ [Takeuchi and Saito, 1972]; $\xi = (\beta_H^2)/(\beta_V^2)$, $\phi = (\alpha_V^2)/(\alpha_H^2)$; ξ and ϕ are S-wave and P-wave anisotropy respectively. Let the waves propagate in x - z plane where x is along horizontal plane and z is vertical (upward). Following Anderson [1961], the displacement components u and w along x and z axes are written as

$$[u, w] = [U(z), W(z)] \exp[ik(ct-x)] \quad (1)$$

where c is phase velocity, k is the wave number and

$$U(z) = A_1 e^{kr_1 z} + A_2 e^{kr_2 z} + B_1 e^{-kr_1 z} + B_2 e^{-kr_2 z} \\ -iW(z) = \gamma_1 A_1 e^{kr_1 z} + \gamma_2 A_2 e^{kr_2 z} - \gamma_1 B_1 e^{-kr_1 z} - \gamma_2 B_2 e^{-kr_2 z} \quad (2)$$

with

$$\gamma_j = \frac{r_j \bar{G}}{\phi \alpha_H^2 r_j^2 - \beta_V^2 + c^2} = -\frac{\beta_V^2 r_j^2 - \alpha_H^2 + c^2}{r_j \bar{G}} \quad (3)$$

$$r_j^2 = (S_1 \pm S_3)/2, \quad j=1, 2 \quad (4)$$

$$S_1 = A + B - C, \quad S_2 = AB, \quad S_3^2 = S_1^2 - 4S_2 \quad (5)$$

$$A = \frac{1}{\beta_V^2} (\alpha_H^2 - c^2), \quad B = \frac{1}{\phi \alpha_H^2} (\beta_V^2 - c^2), \quad C = \frac{1}{\phi} \left[\frac{\eta (\alpha_H^2 - 2\beta_V^2) + \beta_V^2}{\alpha_H \beta_V} \right]^2$$

$$\bar{G} = \eta (\alpha_H^2 - 2\beta_V^2) + \beta_V^2$$

When $S_3^2 \geq 0$, r_j^2 is real making r_j to be either pure real or pure imaginary and is given by

$$r_j = \frac{1}{\sqrt{2}} \sqrt{S_1 \pm S_3}$$

When $S_3^2 < 0$, r_j^2 is complex and given by

$$r_j^2 = \frac{1}{2} \left[S_1 \pm i \sqrt{4S_2 - S_1^2} \right], \quad (6)$$

As given in Equation (5), S_1 is real and so r_j^2 is complex only when $S_3^2 < 0$. Using (5), we can write

$$S_3^2 = (A - B)^2 + C^2 - 2C(A + B)$$

Thus S_3^2 is negative only if $(A+B) > 0$. From (5), $S_2 = (S_1^2 - S_3^2)/4$. Thus S_2 is positive when S_3^2 is negative. Equation (5) shows that S_2 can be positive if A and B are either, both positive or, both negative. Thus to have $(A+B) > 0$, both A and B will be positive and (5) shows that $c < \min[\alpha_H, \beta_V]$. Normally $\beta_V < \alpha_H$, therefore $c < \beta_V$ when S_3^2 is negative.

S_3^2 is a nonlinear function of c/β_V , α_H/β_V , ϕ and η . In a ϕ - η plane the region of negative S_3^2 or complex r_j^2 lies between two nearly parallel lines (Figure 1). The ϕ and η values above complex region are rarely observed. With normal values of ϕ and η in the crust and mantle, complex r_j^2 exists when ϕ is low and η is high. In the subsurface, ϕ may be very low and η may be very high [Xiao 2004]. Strong VTI has often been noted in organic rich shales [Sengupta et al. 2015]. As c/β_V decreases, the com-

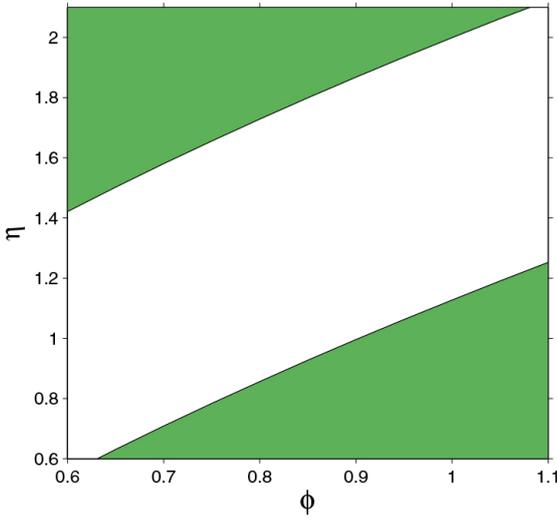


Figure 1. With $\alpha_H/\beta_V = 1.732$ and $c/\beta_V = 0.90$, the green shaded region in $\phi - \eta$ plane shows ϕ and η values for which S_3^2 is positive or r_j^2 is real. White region shows ϕ and η values for which S_3^2 is negative or r_j^2 is complex. ϕ and η are anisotropic parameters [Takeuchi and Saito 1972, Nishimura and Forsyth 1989].

plex region spreads towards both lower and higher anisotropy. As α_H/β_V increases, the complex region is narrower with upper boundary coming down and lower boundary having a minor change.

When $S_3^2 < 0$, r_j^2 as well as r_j is complex and from (6)

$$r_j = \frac{1}{2} \{ \sqrt{[2\sqrt{S_2} + S_1]} \pm i\sqrt{[2\sqrt{S_2} - S_1]} \} = r' \pm ir'' \quad (7a)$$

showing r_1 and r_2 as complex conjugates. In (7a)

$$r' = \frac{1}{2} \sqrt{[2\sqrt{S_2} + S_1]} \text{ and } r'' = \frac{1}{2} \sqrt{[2\sqrt{S_2} - S_1]} \quad (7b)$$

Since $S_3^2 < 0$, (5) shows that $4S_2 > S_1^2$, or $2\sqrt{S_2} > S_1$, because both S_2 and S_1^2 are positive. Thus r' and r'' are real and positive.

In isotropic case $\xi = \phi = \eta = 1$, $\alpha_H = \alpha_V = \alpha$ and $\beta_V = \beta_H = \beta$. Thus $A = (\alpha^2 - c^2)/\beta^2$, $B = (\beta^2 - c^2)/\alpha^2$ and $C = \alpha^2/\beta^2 + \beta^2/\alpha^2 - 2$. (8)

Thus

$$(S_1, S_3) = (1 - c^2/\alpha^2) \pm (1 - c^2/\beta^2) \quad (9)$$

and

$$r_1 = \sqrt{1 - c^2/\alpha^2} = r_\alpha \text{ and } r_2 = \sqrt{1 - c^2/\beta^2} = r_\beta. \quad (10)$$

Further

$$\gamma_1 = r_\alpha \text{ and } \gamma_2 = 1/r_\beta. \quad (11)$$

3. Layer matrix

3.1. Formulation

Here we find the layer matrix of a given layer [Haskell 1953]. The stress components p_{zz} and p_{zx} in terms of $U(z)$ and $W(z)$ are in Anderson [1961]. Thus displacement-stress matrix is written as

$$[Y(z)] = [E(z)][R] \quad (12)$$

where

$$[Y(z)] = [U(z), -iW(z), ip_{zz}/k, p_{zx}/k]^T$$

$$[R] = [A_1 + B_1, A_1 - B_1, A_2 + B_2, A_2 - B_2]^T \quad (13)$$

and

$$[E(z)] = \begin{bmatrix} \cosh \bar{Q}_1 & \sinh \bar{Q}_1 & \cosh \bar{Q}_2 & \sinh \bar{Q}_2 \\ \gamma_1 \sinh \bar{Q}_1 & \gamma_1 \cosh \bar{Q}_1 & \gamma_2 \sinh \bar{Q}_2 & \gamma_2 \cosh \bar{Q}_2 \\ X_1 \cosh \bar{Q}_1 & X_1 \sinh \bar{Q}_1 & X_2 \cosh \bar{Q}_2 & X_2 \sinh \bar{Q}_2 \\ Z_1 \sinh \bar{Q}_1 & Z_1 \cosh \bar{Q}_1 & Z_2 \sinh \bar{Q}_2 & Z_2 \cosh \bar{Q}_2 \end{bmatrix} \quad (14)$$

$$\bar{Q}_j = kr_j z, \quad X_j = c_{13} - c_{33} r_j \gamma_j, \quad Z_j = (r_j + \gamma_j) c_{44} \quad (15)$$

$$c_{13} = \eta \rho (\alpha_H^2 - 2\beta_V^2), \quad c_{33} = \rho \phi \alpha_H^2, \quad c_{44} = \rho \beta_V^2 \quad (16)$$

Evaluating $Y(-H)$ and $Y(0)$ from (12) and eliminating $[R]$ we get

$$[Y(-H)] = [E(-H)][E(0)]^{-1}[Y(0)] = [a][Y(0)] \quad (17)$$

Here H is the thickness of the layer, $[Y(-H)]$ is the displacement-stress matrix at the bottom of the layer and $[Y(0)]$ is that at the top. In (17), $[a]$ is the layer matrix given by

$$[a] = [E(-H)][E(0)]^{-1} \quad (18)$$

where from (14)

$$[E(-H)] = \begin{bmatrix} \cosh Q_1 & -\sinh Q_1 & \cosh Q_2 & -\sinh Q_2 \\ -\gamma_1 \sinh Q_1 & \gamma_1 \cosh Q_1 & -\gamma_2 \sinh Q_2 & \gamma_2 \cosh Q_2 \\ X_1 \cosh Q_1 & -X_1 \sinh Q_1 & X_2 \cosh Q_2 & -X_2 \sinh Q_2 \\ -Z_1 \sinh Q_1 & Z_1 \cosh Q_1 & -Z_2 \sinh Q_2 & Z_2 \cosh Q_2 \end{bmatrix}$$

$$Q_j = kr_j H,$$

$$[E(0)]^{-1} = \begin{bmatrix} -X_2/X_{12} & 0 & 1/X_{12} & 0 \\ 0 & Z_2/\Gamma_{12} & 0 & -\gamma_2/\Gamma_{12} \\ X_1/X_{12} & 0 & -1/X_{12} & 0 \\ 0 & -Z_1/\Gamma_{12} & 0 & \gamma_1/\Gamma_{12} \end{bmatrix} \quad (19)$$

$$X_{12} = X_1 - X_2 \quad \text{and} \quad \Gamma_{12} = \gamma_1 Z_2 - \gamma_2 Z_1 \quad (20)$$

Thus from (18)

$[a] =$

$$\begin{bmatrix} \frac{1}{X_{12}}(-X_2 \cosh Q_1 + X_1 \cosh Q_2) & \frac{1}{\Gamma_{12}}(-Z_2 \sinh Q_1 + Z_1 \sinh Q_2) & \frac{1}{X_{12}}(\cosh Q_1 - \cosh Q_2) & \frac{1}{\Gamma_{12}}(\gamma_2 \sinh Q_1 - \gamma_1 \sinh Q_2) \\ \frac{1}{X_{12}}(\gamma_1 X_2 \sinh Q_1 - \gamma_2 X_1 \sinh Q_2) & \frac{1}{\Gamma_{12}}(\gamma_1 Z_2 \cosh Q_1 - \gamma_2 Z_1 \cosh Q_2) & \frac{1}{X_{12}}(-\gamma_1 \sinh Q_1 + \gamma_2 \sinh Q_2) & \frac{\gamma_2 Z_2}{\Gamma_{12}}(-\cosh Q_1 + \cosh Q_2) \\ \frac{X_2 Z_2}{X_{12}}(-\cosh Q_1 + \cosh Q_2) & \frac{1}{\Gamma_{12}}(-X_1 Z_2 \sinh Q_1 + X_2 Z_1 \sinh Q_2) & \frac{1}{X_{12}}(X_1 \cosh Q_1 - X_2 \cosh Q_2) & \frac{1}{\Gamma_{12}}(X_2 \gamma_1 \sinh Q_1 - X_1 \gamma_2 \sinh Q_2) \\ \frac{1}{X_{12}}(Z_1 X_2 \sinh Q_1 - Z_2 X_1 \sinh Q_2) & \frac{Z_2 Z_1}{\Gamma_{12}}(\cosh Q_1 - \cosh Q_2) & \frac{1}{X_{12}}(-Z_2 \sinh Q_1 + Z_1 \sinh Q_2) & \frac{1}{\Gamma_{12}}(-Z_2 \gamma_1 \cosh Q_1 + Z_1 \gamma_2 \cosh Q_2) \end{bmatrix} \quad (21)$$

3.2 Case I: $S_3^2 \geq 0$

In this case, r_j is either real or imaginary. We shall rearrange the matrix elements of $[a]$ in (21) so that each element becomes real even if r_j is imaginary. Let us denote

$$g_j = \frac{\tilde{G}}{\phi \alpha_H^2 r_j^2 - \beta_V^2 + c^2} \quad (22)$$

Thus from (3), $\gamma_j = r_j g_j$. From (15), we have

$$X_j = c_{13} - c_{33} r_j^2 g_j \quad \text{and} \quad Z_j = c_{44} r_j (1 + g_j) \quad (23)$$

It may be further noted that

$$\Gamma_{12} = c_{44} r_1 r_2 (g_1 - g_2) = c_{44} r_1 r_2 g_{12} \quad (24)$$

where $g_{12} = g_1 - g_2$. Now we write the elements of layer matrix $[a]$ as

$$\begin{aligned} a_{11} &= \frac{1}{x_{12}} (-X_2 \cosh Q_1 + X_1 \cosh Q_2), \\ a_{12} &= \frac{1}{g_{12}} \left[-(1 + g_2) \frac{\sinh Q_1}{r_1} + (1 + g_1) \frac{\sinh Q_2}{r_2} \right], \\ a_{13} &= \frac{1}{x_{12}} (\cosh Q_1 - \cosh Q_2), \\ a_{14} &= \frac{1}{c_{44} g_{12}} \left[g_2 \frac{\sinh Q_1}{r_1} - g_1 \frac{\sinh Q_2}{r_2} \right], \\ a_{21} &= \frac{1}{x_{12}} (X_2 g_1 r_1 \sinh Q_1 - X_1 g_2 r_2 \sinh Q_2), \\ a_{22} &= \frac{1}{g_{12}} [g_1 (1 + g_2) \cosh Q_1 - g_2 (1 + g_1) \cosh Q_2], \\ a_{23} &= \frac{1}{x_{12}} (-g_1 r_1 \sinh Q_1 + g_2 r_2 \sinh Q_2), \\ a_{24} &= \frac{g_1 g_2}{c_{44} g_{12}} (-\cosh Q_1 + \cosh Q_2), \\ a_{31} &= \frac{x_1 x_2}{x_{12}} (-\cosh Q_1 + \cosh Q_2), \\ a_{32} &= \frac{1}{g_{12}} \left[-X_1 (1 + g_2) \frac{\sinh Q_1}{r_1} + X_2 (1 + g_1) \frac{\sinh Q_2}{r_2} \right], \\ a_{33} &= \frac{1}{x_{12}} (X_1 \cosh Q_1 - X_2 \cosh Q_2), \\ a_{34} &= \frac{1}{c_{44} g_{12}} \left[X_1 g_2 \frac{\sinh Q_1}{r_1} - X_2 g_1 \frac{\sinh Q_2}{r_2} \right], \\ a_{41} &= \frac{c_{44}}{x_{12}} [X_2 (1 + g_1) r_1 \sinh Q_1 - X_1 (1 + g_2) r_2 \sinh Q_2], \\ a_{42} &= \frac{c_{44}}{g_{12}} (1 + g_1) (1 + g_2) (\cosh Q_1 - \cosh Q_2), \\ a_{43} &= \frac{c_{44}}{x_{12}} [-(1 + g_1) r_1 \sinh Q_1 + (1 + g_2) r_2 \sinh Q_2] \\ a_{44} &= \frac{1}{g_{12}} [-g_2 (1 + g_1) \cosh Q_1 + g_1 (1 + g_2) \cosh Q_2]. \end{aligned} \quad (25)$$

Similar elements were also derived by Harkrider and Anderson [1962]. Note that $\cosh Q_j$, $(\sinh Q_j)/r_j$ and $r_j \sinh Q_j$ are real when r_j is imaginary.

3.3 Case II: $S_3^2 < 0$

In this case r_1 and r_2 are complex conjugates and so Q_1 and Q_2 are also complex conjugates and we denote

$$Q_1 = Q' + iQ'' \quad \text{and} \quad Q_2 = Q' - iQ'' \quad (26)$$

where $Q' = kr'H$ and $Q'' = kr''H$. Substituting in (21) and denoting

$$\begin{aligned} F_1 &= \cosh Q' \cos Q'', & F_2 &= \cosh Q' \sin Q'', \\ F_3 &= \sinh Q' \sin Q'', & F_4 &= \sinh Q' \cos Q'', \end{aligned} \quad (27)$$

we get the elements of the layer matrix as

$$\begin{aligned} a_{11} &= F_1 - i \frac{x_1 + x_2}{x_{12}} F_3, \\ a_{12} &= -i \frac{z_1 + z_2}{\Gamma_{12}} F_2 + \frac{z_1 - z_2}{\Gamma_{12}} F_4, \\ a_{13} &= i \frac{2}{x_{12}} F_3, \\ a_{14} &= i \frac{\gamma_1 + \gamma_2}{\Gamma_{12}} F_2 - \frac{\gamma_1 - \gamma_2}{\Gamma_{12}} F_4, \\ a_{21} &= i \frac{x_1 \gamma_2 + x_2 \gamma_1}{x_{12}} F_2 - \frac{x_1 \gamma_2 - x_2 \gamma_1}{x_{12}} F_4, \\ a_{22} &= F_1 + i \frac{\gamma_1 z_2 + \gamma_2 z_1}{\Gamma_{12}} F_3, \\ a_{23} &= -i \frac{\gamma_1 + \gamma_2}{x_{12}} F_2 - \frac{\gamma_1 - \gamma_2}{x_{12}} F_4, \\ a_{24} &= -i \frac{2\gamma_1 \gamma_2}{\Gamma_{12}} F_3, \\ a_{31} &= -i \frac{2x_1 x_2}{x_{12}} F_3, \\ a_{32} &= -i \frac{x_1 z_2 + x_2 z_1}{\Gamma_{12}} F_2 - \frac{x_1 z_2 - x_2 z_1}{\Gamma_{12}} F_4, \\ a_{33} &= F_1 + i \frac{x_1 + x_2}{x_{12}} F_3, \\ a_{34} &= i \frac{x_1 \gamma_2 + x_2 \gamma_1}{\Gamma_{12}} F_2 + \frac{x_1 \gamma_2 - x_2 \gamma_1}{\Gamma_{12}} F_4, \\ a_{41} &= i \frac{x_1 z_2 + x_2 z_1}{x_{12}} F_2 - \frac{x_1 z_2 - x_2 z_1}{x_{12}} F_4, \\ a_{42} &= i \frac{2z_1 z_2}{\Gamma_{12}} F_3, \\ a_{43} &= -i \frac{z_1 + z_2}{x_{12}} F_2 - \frac{z_1 - z_2}{x_{12}} F_4, \\ a_{44} &= F_1 - i \frac{\gamma_1 z_2 + \gamma_2 z_1}{\Gamma_{12}} F_3. \end{aligned} \quad (28)$$

The pairs (r_1, r_1) , (γ_1, γ_2) , (X_1, X_2) , (Z_2, Z_2) etc. are complex conjugates. Sum of the conjugates is real and difference is imaginary. Further multiplication of complex conjugates is real. Thus each element of (28) is real.

4. Dispersion equation

4.1 Solid layers

Let us consider N homogeneous VTI layers numbering 1 to N from top to bottom, the N th layer being the homogeneous half space. The m th layer is bounded by $(m-1)$ th interface at the top and m th interface at the bottom. We use suffix m to show the values in the m th layer and superscript m in bracket to show the values at the interface m .

The constants $(B_1)_N$ and $(B_2)_N$ are zero since displacement is zero at infinite depth $z = -\infty$. From (12), we have at the top of N th layer

$$[Y_N(0)]^{(N-1)} = [E_N(0)] [A_1, A_1, A_2, A_2]_N^T$$

or

$$[A_1, A_1, A_2, A_2]_N^T = [E_N(0)]^{-1} [Y_N(0)]^{(N-1)} \quad (29)$$

Following Haskell [1953], we use continuity of displacement-stress matrix at interfaces and use the relation (17) to obtain

$$\begin{aligned} [A_1, A_1, A_2, A_2]_N^T &= [E_N(0)]^{-1} [a_{N-1}] \dots \dots [a_2][a_1][Y_1(0)] \\ &= [E_N(0)]^{-1} [M] [Y_1(0)] = [J][Y_1(0)] \end{aligned} \quad (30)$$

where the displacement-stress matrix $[Y_1(0)]$ at the free surface is

$$[Y_1(0)] = [U_0, -iW_0, 0, 0]^T \quad (31)$$

with $U_0 = U(0)$ and $W_0 = W(0)$. In (30)

$$[M] = [a_{N-1}] \dots \dots [a_2][a_1]$$

and

$$[J] = [E_N(0)]^{-1} [M]. \quad (32)$$

Equating the rows in (30) and eliminating U_0 , W_0 , A_{1N} and A_{2N} , we get the dispersion equation as

$$\frac{J_{11} - J_{21}}{J_{31} - J_{41}} = \frac{J_{12} - J_{22}}{J_{32} - J_{42}} \quad (33)$$

Multiplying both sides of (32) by

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

and using (19) we get

$$\begin{bmatrix} J_{11} - J_{21} & J_{12} - J_{22} & J_{13} - J_{23} & J_{14} - J_{24} \\ J_{31} - J_{41} & J_{32} - J_{42} & J_{33} - J_{43} & J_{34} - J_{44} \end{bmatrix} = \begin{bmatrix} -\frac{X_2}{X_{12}} & -\frac{Z_2}{\Gamma_{12}} & \frac{1}{X_{12}} & \frac{Y_2}{\Gamma_{12}} \\ \frac{X_1}{X_{12}} & \frac{Z_1}{\Gamma_{12}} & -\frac{1}{X_{12}} & -\frac{Y_1}{\Gamma_{12}} \end{bmatrix}_N [M] \quad (34)$$

The dispersion equation (33) may be written as

$$(J_{11} - J_{21})(J_{32} - J_{42}) - (J_{12} - J_{22})(J_{31} - J_{41}) = 0 \quad (35)$$

Equation (30) also gives surface ellipticity ε as

$$\varepsilon = \frac{U_0}{-iW_0} = -\frac{J_{12} - J_{22}}{J_{11} - J_{21}} \quad (36)$$

4.2 Liquid layer at the top

Let us consider a liquid layer of density ρ_0 , P-wave velocity α_0 and thickness H_0 over the above N -layered VTI half space. The vertical components of displacement and stress in the liquid layer are respectively

$$W(z) = kr_0 C_0 \cos(kr_0 z)$$

$$p_{zz}(z) = -\rho \omega^2 C_0 \sin(kr_0 z)$$

where $r_0 = \sqrt{c^2/\alpha_0^2 - 1}$ [Ewing et al. 1957]. Thus in equation (30) instead of displacement-stress matrix $[Y_1(0)]$ on free surface, we use the displacement-stress matrix at the bottom ($z = -H_0$) of the liquid layer as

$$Y_1(-H_0) = [U_0, -ikr_0 C_0 \cos(kr_0 H_0), iC_0(\omega^2 \rho/k) \sin(kr_0 H_0), 0]^T$$

Here U_0 is the horizontal displacement component in the top solid layer which is just below the liquid layer. As in section 4.1, equating the rows in equation (30) and eliminating U_0 , C_0 , A_{1N} and A_{2N} we get the dispersion equation as

$$\frac{J_{12} - J_{22}}{J_{11} - J_{21}} - \frac{J_{32} - J_{42}}{J_{31} - J_{41}} - \chi \left[\frac{J_{13} - J_{23}}{J_{11} - J_{21}} - \frac{J_{33} - J_{43}}{J_{31} - J_{41}} \right] = 0 \quad (37)$$

where

$$\chi = \frac{\omega^2 \rho}{k^2 r_0} \tan(kr_0 H_0) = \frac{\rho c^2}{r_0} \tan(kr_0 H_0) \quad (38)$$

Similar dispersion in isotropic layered half space with same χ was obtained by Harkrider (1964). The dispersion equation (37) can be written as

$$\begin{aligned} (J_{11} - J_{21})(J_{32} - J_{42}) - (J_{12} - J_{22})(J_{31} - J_{41}) \\ - \chi [(J_{11} - J_{21})(J_{33} - J_{43}) - (J_{13} - J_{23})(J_{31} - J_{41})] = 0 \end{aligned} \quad (39)$$

If liquid layer is absent $H_0 = 0$ and so $\chi = 0$; thus equation (39) has the same form as (35). As in section 4.1, the surface ellipticity at the top of solid layers is given by

$$\varepsilon = \frac{U_0}{-iW_0} = -\frac{J_{12} - J_{22}}{J_{11} - J_{21}} + \chi \frac{J_{13} - J_{23}}{J_{11} - J_{21}} \quad (40)$$

5. Delta matrix

We shall now obtain delta (compound) matrix [Throrer 1965, Harkrider 1970] of the layer matrix derived in section 3. We shall use a bar to denote a delta matrix e.g. delta matrix of $[a]$ is $[\bar{a}]$.

5.1 Case I: $S_3^2 \geq 0$

Denoting

$$Ch_j = \cosh Q_j \text{ and } Sh_j = (\sinh Q_j) / r_j$$

the elements of the delta matrix $[\bar{a}]$ obtained from the matrix (25) are as follows.

$$\bar{a}_{11} = \{-X_1(1+g_1)g_2 - X_2(1+g_2)g_1 + Ch_1Ch_2[X_1g_1(1+g_2) + X_2g_2(1+g_1)] - Sh_1Sh_2[X_1g_2(1+g_2)r_2^2 + X_2g_1(1+g_1)r_1^2]\} / (X_{12}g_{12})$$

$$\bar{a}_{12} = \{g_2r_2^2Ch_1Sh_2 - g_1r_1^2Ch_2Sh_1\} / X_{12}$$

$$\bar{a}_{13} = \{g_1g_2(X_1+X_2)(1-Ch_1Ch_2) + Sh_1Sh_2[X_1g_2^2r_2^2 + X_2g_1^2r_1^2]\} / (c_{44}g_{12}X_{12})$$

$$\bar{a}_{14} = \{-(g_1+g_2+2g_1g_2)(1-Ch_1Ch_2) - Sh_1Sh_2[(1+g_2)g_2r_2^2 + (1+g_1)g_1r_1^2]\} / (X_{12}g_{12}) = \bar{a}_{13}$$

$$\bar{a}_{15} = \{Ch_1Sh_2g_1 - Ch_2Sh_1g_2\} / (c_{44}g_{12})$$

$$\bar{a}_{16} = \{-g_1g_2(Ch_1 - Ch_2)^2 + (g_2Sh_1 - g_1Sh_2)(g_1r_1^2Sh_1 - g_2r_2^2Sh_2)\} / (c_{44}g_{12}X_{12})$$

$$\bar{a}_{21} = \{(1+g_1)X_2Ch_1Sh_2 - (1+g_2)X_1Ch_2Sh_1\} / g_{12}, \quad \bar{a}_{22} = Ch_1Ch_2$$

$$\bar{a}_{23} = \{X_1g_2Ch_2Sh_1 - X_2g_1Ch_1Sh_2\} / (c_{44}g_{12}),$$

$$\bar{a}_{24} = \{(1+g_1)Ch_1Sh_2 - (1+g_2)Ch_2Sh_1\} / g_{12} = \bar{a}_{23},$$

$$\bar{a}_{25} = -X_{12}Sh_1Sh_2 / (c_{44}g_{12}), \quad \bar{a}_{26} = \bar{a}_{15},$$

$$\bar{a}_{31} = \{-(1+g_1)(1+g_2)(X_1+X_2)(1-Ch_1Ch_2)$$

$$-Sh_1Sh_2[(1+g_1)^2r_1^2X_2 + (1+g_2)^2r_2^2X_1]c_{44}\} / (X_{12}g_{12})$$

$$\bar{a}_{32} = \{-(1+g_1)r_1^2Ch_2Sh_1 + (1+g_2)r_2^2Ch_1Sh_2\}c_{44} / X_{12}$$

$$\bar{a}_{33} = \{X_1g_1(1+g_2) + X_2g_2(1+g_1) - Ch_1Ch_2[X_1(1+g_1)g_2 + X_2(1+g_2)g_1 + Sh_1Sh_2[X_1g_2r_2^2(1+g_2) + X_2g_1r_1^2(1+g_1)]]\} / (X_{12}g_{12})$$

$$\bar{a}_{34} = \{-2(1-Ch_1Ch_2)(1+g_1)(1+g_2)\}$$

$$-Sh_1Sh_2[(1+g_1)^2r_1^2 + (1+g_2)^2r_2^2]c_{44}\} / (X_{12}g_{12}) = \bar{a}_{33} - 1$$

$$\bar{a}_{35} = \bar{a}_{24}, \quad \bar{a}_{36} = \bar{a}_{14}$$

$$\bar{a}_{41} = \{X_1X_2[g_1(1+g_2) + g_2(1+g_1)](1-Ch_1Ch_2)$$

$$+ Sh_1Sh_2[X_1^2g_2(1+g_2)r_2^2 + X_2^2g_1(1+g_1)r_1^2]\} / (X_{12}g_{12}) = \bar{a}_{31}$$

$$\bar{a}_{42} = \{g_1r_1^2X_2Ch_2Sh_1 - g_2r_2^2X_1Ch_1Sh_2\} / X_{12} = \bar{a}_{32}$$

$$\bar{a}_{43} = \{-2X_1X_2g_1g_2(1-Ch_1Ch_2) - Sh_1Sh_2[X_1^2g_2^2r_2^2 + X_2^2g_1^2r_1^2]\} / (c_{44}g_{12}X_{12}) = \bar{a}_{34}$$

$$\bar{a}_{44} = \bar{a}_{33}, \quad \bar{a}_{45} = \bar{a}_{23}, \quad \bar{a}_{46} = \bar{a}_{13},$$

$$\bar{a}_{51} = \{X_1r_2^2(1+g_2)Ch_1Sh_2 - X_2r_1^2(1+g_1)Ch_2Sh_1\}c_{44} / X_{12}$$

$$\bar{a}_{52} = \{-r_1^2r_2^2g_{12}Sh_1Sh_2\}c_{44} / X_{12}$$

$$\bar{a}_{53} = \bar{a}_{42}, \quad \bar{a}_{54} = \bar{a}_{32}, \quad \bar{a}_{55} = \bar{a}_{22}, \quad \bar{a}_{56} = \bar{a}_{12},$$

$$\begin{aligned} \bar{a}_{61} &= \{-2X_1X_2(1+g_1)(1+g_2)(1-Ch_1Ch_2) \\ &- Sh_1Sh_2[X_1^2(1+g_2)^2r_2^2 + X_2^2(1+g_1)^2r_1^2]\}c_{44} / (X_{12}g_{12}), \\ \bar{a}_{62} &= \bar{a}_{51}, \quad \bar{a}_{63} = \bar{a}_{41}, \quad \bar{a}_{64} = \bar{a}_{31}, \quad \bar{a}_{65} = \bar{a}_{21}, \quad \bar{a}_{66} = \bar{a}_{11}. \end{aligned} \quad (41)$$

5.2 Case II: $S_3^2 < 0$

We have already defined F_n in equations (27) for $n = 1, 2, 3, 4$. Let us further define

$$F_5 = F_1F_4 + F_2F_3 = \cosh Q' \sinh Q', \quad F_6 = F_1F_2 - F_3F_4 = \cos Q'' \sin Q''$$

It may be noted that

$$F_2F_4 - F_1F_3 = 0.$$

Thus from (28), the elements of delta matrix $[\bar{a}]$ are written as

$$\bar{a}_{11} = F_1^2 - \frac{Z_1+Z_2}{\Gamma_{12}} \frac{X_1\gamma_2 + X_2\gamma_1}{X_{12}} F_2^2 + \frac{X_1+X_2}{X_{12}} \frac{\gamma_1Z_2 + \gamma_2Z_1}{\Gamma_{12}} F_3^2 + \frac{Z_1-Z_2}{\Gamma_{12}} \frac{X_1\gamma_2 - X_2\gamma_1}{X_{12}} F_4^2$$

$$\bar{a}_{12} = -\frac{\gamma_1 - \gamma_2}{X_{12}} F_5 - i \frac{\gamma_1 + \gamma_2}{X_{12}} F_6$$

$$\bar{a}_{13} = \frac{\gamma_1 + \gamma_2}{\Gamma_{12}} \frac{X_1\gamma_2 + X_2\gamma_1}{X_{12}} F_2^2 - \frac{X_1+X_2}{X_{12}} \frac{2\gamma_1\gamma_2}{\Gamma_{12}} F_3^2 - \frac{\gamma_1 - \gamma_2}{\Gamma_{12}} \frac{X_1\gamma_2 - X_2\gamma_1}{X_{12}} F_4^2$$

$$\bar{a}_{14} = -\frac{\gamma_1 + \gamma_2}{X_{12}} \frac{Z_1+Z_2}{\Gamma_{12}} F_2^2 + \frac{2(\gamma_1Z_2 + \gamma_2Z_1)}{X_{12}\Gamma_{12}} F_3^2 - \frac{\gamma_1 - \gamma_2}{X_{12}} \frac{Z_1-Z_2}{\Gamma_{12}} F_4^2 = \bar{a}_{13}$$

$$\bar{a}_{15} = \frac{\gamma_1 - \gamma_2}{\Gamma_{12}} F_5 - i \frac{\gamma_1 + \gamma_2}{\Gamma_{12}} F_6,$$

$$\bar{a}_{16} = -\frac{(\gamma_1 + \gamma_2)^2}{X_{12}\Gamma_{12}} F_2^2 + \frac{4\gamma_1\gamma_2}{X_{12}\Gamma_{12}} F_3^2 - \frac{(\gamma_1 - \gamma_2)^2}{X_{12}\Gamma_{12}} F_4^2,$$

$$\bar{a}_{21} = -\frac{X_1Z_2 - X_2Z_1}{\Gamma_{12}} F_5 - i \frac{X_1Z_2 + X_2Z_1}{\Gamma_{12}} F_6,$$

$$\bar{a}_{22} = F_1^2 + F_3^2,$$

$$\bar{a}_{23} = \frac{X_1\gamma_2 - X_2\gamma_1}{\Gamma_{12}} F_5 + i \frac{X_1\gamma_2 + X_2\gamma_1}{\Gamma_{12}} F_6,$$

$$\bar{a}_{24} = \frac{Z_1 - Z_2}{\Gamma_{12}} F_5 - i \frac{Z_1 + Z_2}{\Gamma_{12}} F_6 = \bar{a}_{23},$$

$$\bar{a}_{25} = -\frac{X_1 - X_2}{\Gamma_{12}} (F_2^2 + F_4^2), \quad \bar{a}_{26} = \bar{a}_{15},$$

$$\bar{a}_{31} = -\frac{Z_1+Z_2}{\Gamma_{12}} \frac{X_1Z_2 + X_2Z_1}{X_{12}} F_2^2 + \frac{2Z_1Z_2(X_1+X_2)}{X_{12}\Gamma_{12}} F_3^2 + \frac{Z_1-Z_2}{\Gamma_{12}} \frac{X_1Z_2 - X_2Z_1}{X_{12}} F_4^2,$$

$$\bar{a}_{32} = -\frac{Z_1 - Z_2}{X_{12}} F_5 - i \frac{Z_1 + Z_2}{X_{12}} F_6,$$

$$\bar{a}_{33} = F_1^2 + \frac{\gamma_1 + \gamma_2}{\Gamma_{12}} \frac{X_1Z_2 + X_2Z_1}{X_{12}} F_2^2 - \frac{X_1+X_2}{X_{12}} \frac{\gamma_1Z_2 + \gamma_2Z_1}{\Gamma_{12}} F_3^2 - \frac{\gamma_1 - \gamma_2}{\Gamma_{12}} \frac{X_1Z_2 - X_2Z_1}{X_{12}} F_4^2,$$

$$\bar{a}_{34} = -\frac{(Z_1+Z_2)^2}{X_{12}\Gamma_{12}} F_2^2 + \frac{4Z_1Z_2}{X_{12}\Gamma_{12}} F_3^2 - \frac{(Z_1-Z_2)^2}{X_{12}\Gamma_{12}} F_4^2,$$

$$\bar{a}_{35} = \bar{a}_{24}, \quad \bar{a}_{36} = \bar{a}_{14},$$

$$\bar{a}_{41} = \frac{X_1\gamma_2 + X_2\gamma_1}{X_{12}} \frac{X_1Z_2 + X_2Z_1}{\Gamma_{12}} F_2^2 - \frac{2X_1X_2}{X_{12}} \frac{\gamma_1Z_2 + \gamma_2Z_1}{\Gamma_{12}} F_3^2 + \frac{X_1\gamma_2 - X_2\gamma_1}{X_{12}} \frac{X_1Z_2 - X_2Z_1}{\Gamma_{12}} F_4^2 = \bar{a}_{31}.$$

$$\bar{a}_{42} = -\frac{X_1\gamma_2 - X_2\gamma_1}{X_{12}} F_5 + i \frac{X_1\gamma_2 + X_2\gamma_1}{X_{12}} F_6 = \bar{a}_{32},$$

$$\bar{a}_{43} = -\frac{(X_1\gamma_2 + X_2\gamma_1)^2}{X_{12}\Gamma_{12}} F_2^2 + 4 \frac{\gamma_1\gamma_2}{\Gamma_{12}} \frac{X_1X_2}{X_{12}} F_3^2 - \frac{(X_1\gamma_2 - X_2\gamma_1)^2}{X_{12}\Gamma_{12}} F_4^2 = \bar{a}_{34},$$

$$\begin{aligned}
 \bar{a}_{44} &= \bar{a}_{33}, & \bar{a}_{45} &= \bar{a}_{23}, & \bar{a}_{46} &= \bar{a}_{13}, \\
 \bar{a}_{51} &= \frac{X_1 Z_2 - X_2 Z_1}{X_{12}} F_5 - i \frac{X_1 Z_2 + X_2 Z_1}{X_{12}} F_6, & \bar{a}_{52} &= -\frac{\Gamma_{12}}{X_{12}} (F_4^2 + F_2^2), \\
 \bar{a}_{53} &= \bar{a}_{42}, & \bar{a}_{54} &= \bar{a}_{32}, & \bar{a}_{55} &= \bar{a}_{22}, & \bar{a}_{56} &= \bar{a}_{12}, \\
 \bar{a}_{61} &= -\frac{(X_1 Z_2 + X_2 Z_1)^2}{X_{12} \Gamma_{12}} F_2^2 + 4 \frac{X_1 X_2 Z_1 Z_2}{X_{12} \Gamma_{12}} F_3^2 - \frac{(X_1 Z_2 - X_2 Z_1)^2}{X_{12} \Gamma_{12}} F_4^2, \\
 \bar{a}_{62} &= \bar{a}_{51}, & \bar{a}_{63} &= \bar{a}_{41}, & \bar{a}_{64} &= \bar{a}_{31}, & \bar{a}_{65} &= \bar{a}_{21}, & \bar{a}_{66} &= \bar{a}_{11}.
 \end{aligned} \tag{42}$$

Based on the arguments given below Equation (28), each term in each of the above matrix element is real.

6. Dispersion equation in terms of delta matrix

We write the equation (34) as

$$[K] = [P] [M] \tag{43}$$

where

$$[K] = \begin{bmatrix} J_{11} - J_{21} & J_{12} - J_{22} & J_{13} - J_{23} & J_{14} - J_{24} \\ J_{31} - J_{41} & J_{32} - J_{42} & J_{33} - J_{43} & J_{34} - J_{44} \end{bmatrix},$$

and

$$[P] = \begin{bmatrix} -\frac{X_2}{X_{12}} & -\frac{Z_2}{\Gamma_{12}} & \frac{1}{X_{12}} & \frac{Y_2}{\Gamma_{12}} \\ \frac{X_1}{X_{12}} & \frac{Z_1}{\Gamma_{12}} & -\frac{1}{X_{12}} & -\frac{Y_1}{\Gamma_{12}} \end{bmatrix}_N$$

Using the properties of delta matrix, from (43) we get

$$[\bar{K}] = [\bar{P}] [\bar{M}] \tag{44}$$

where $[\bar{K}]$ and $[\bar{P}]$ are (1×6) matrices with elements

$$\bar{K}_{11} = (J_{11} - J_{21})(J_{32} - J_{42}) - (J_{12} - J_{22})(J_{31} - J_{41}),$$

$$\bar{K}_{12} = (J_{11} - J_{21})(J_{33} - J_{43}) - (J_{13} - J_{23})(J_{31} - J_{41}), \text{ etc.}$$

$$[\bar{P}] = \left[\frac{X_1 Z_2 - X_2 Z_1}{X_{12} \Gamma_{12}}, \frac{1}{X_{12}}, -\frac{X_1 Y_2 - X_2 Y_1}{X_{12} \Gamma_{12}}, -\frac{Z_1 - Z_2}{X_{12} \Gamma_{12}}, \frac{1}{\Gamma_{12}}, -\frac{Y_1 - Y_2}{X_{12} \Gamma_{12}} \right],$$

and

$$[\bar{M}] = [\bar{a}_{N-1}] [\bar{a}_{N-2}] \dots \dots [\bar{a}_3] [\bar{a}_2] [\bar{a}_1].$$

The dispersion equation (39) is written as

$$\bar{K}_{11} - \chi \bar{K}_{12} = 0$$

where $\chi = 0$ when top liquid layer is absent i.e. when $H_0 = 0$. Using (44), the above dispersion equation can be written as

$$[\bar{P}] [\bar{a}_{N-1}] [\bar{a}_{N-2}] \dots \dots [\bar{a}_3] [\bar{a}_2] [\hat{a}_1] = 0 \tag{45}$$

where $[\hat{a}_1]$ is a (6×1) matrix given by

$$[\hat{a}_1] = [\bar{a}_{11} - \chi \bar{a}_{12}, \bar{a}_{21} - \chi \bar{a}_{22}, \bar{a}_{31} - \chi \bar{a}_{32}, \bar{a}_{41} - \chi \bar{a}_{42}, \bar{a}_{51} - \chi \bar{a}_{52}, \bar{a}_{61} - \chi \bar{a}_{62}]^T$$

If r_j is complex in the N th layer, each term of $[\bar{P}]$ has a factor i , which gets eliminated in equation (45). $[\bar{a}_m]$ has already been seen as real ($m = 1, 2, \dots, N-1$). Thus the dispersion equation (45) contains all real terms.

Relationships among the elements of delta matrix $[\bar{a}]$ and $\bar{P}_{13} = \bar{P}_{14}$ show that $\bar{K}_{13} = \bar{K}_{14}$. Following Harkrider [1964, Equation 74], we write surface ellipticity ε from equation (36) as

$$\varepsilon = -\bar{K}_{13} / \bar{K}_{12} \tag{46}$$

which is real even if r_j is complex. The above expression for ε is also obtained from equation (40) at the top of solid layers below the liquid layer.

7. Dispersion equation in terms of reduced delta matrix

The relationship among the elements of $[\bar{a}_m]$ have been indicated in equations (41) and (42); further $\bar{P}_{13} = \bar{P}_{14}$. Thus the dispersion equation (45) can be written as

$$[\bar{P}] [\bar{a}_{N-1}] [\bar{a}_{N-2}] \dots \dots [\bar{a}_3] [\bar{a}_2] [\hat{a}_1] = 0 \tag{47}$$

where the reduced delta matrix $[\bar{a}_m]$ of m th layer is given by

$$[\bar{a}_m] = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & 2\bar{a}_{13} & \bar{a}_{15} & \bar{a}_{16} \\ \bar{a}_{21} & \bar{a}_{22} & 2\bar{a}_{23} & \bar{a}_{25} & \bar{a}_{26} \\ \bar{a}_{31} & \bar{a}_{32} & \bar{a}_{33} + \bar{a}_{34} & \bar{a}_{35} & \bar{a}_{36} \\ \bar{a}_{51} & \bar{a}_{52} & 2\bar{a}_{53} & \bar{a}_{55} & \bar{a}_{56} \\ \bar{a}_{61} & \bar{a}_{62} & 2\bar{a}_{63} & \bar{a}_{65} & \bar{a}_{66} \end{bmatrix}_m$$

In equation (47)

$$[\bar{P}] = [\bar{P}_{11}, \bar{P}_{12}, 2\bar{P}_{13}, \bar{P}_{15}, \bar{P}_{16}]$$

and

$$[\hat{a}_1] = [\bar{a}_{11} - \chi \bar{a}_{12}, \bar{a}_{21} - \chi \bar{a}_{22}, \bar{a}_{31} - \chi \bar{a}_{32}, \bar{a}_{41} - \chi \bar{a}_{42}, \bar{a}_{51} - \chi \bar{a}_{52}]^T$$

In terms of reduced layer matrix surface ellipticity ε in equation (46) can be written as

$$\varepsilon = -0.5 \bar{K}_3 / \bar{K}_2 \tag{48}$$

where

$$\bar{K}_j = [\bar{P}] [\bar{a}_{N-1}] [\bar{a}_{N-2}] \dots \dots [\bar{a}_3] [\bar{a}_2] [\hat{a}_{1(j)}]$$

with $j = 2, 3$ and $[\hat{a}_{1(j)}]$ is the j th column of $[\hat{a}_1]$.

8. Computational results and discussions

Based on dispersion equation (47), a FORTRAN code in real domain is prepared to evaluate phase and

group velocities; surface ellipticity of Rayleigh waves is obtained using equation (48). Considering dispersion equation (47) as $f(k,c)=0$, group velocity (U) is obtained using

$$U = c - k \frac{(\partial f / \partial k)_c}{(\partial f / \partial c)_k}$$

where the above partial derivatives in the numerator and denominator are obtained analytically from f . The code may be shared on request. Using this code, dispersion curves are drawn here in Figures 2, 3 and 5 for a few existing oceanic and continental VTI models. Through generalized Reflection-Transmission method, Bhattacharya and Arora [1997] obtained dispersion equation with complex terms in a layered VTI spherical earth using a flattening transformation. This method to get dispersion curves in a layered VTI half space works in a complex domain and gives the same results as from the present code.

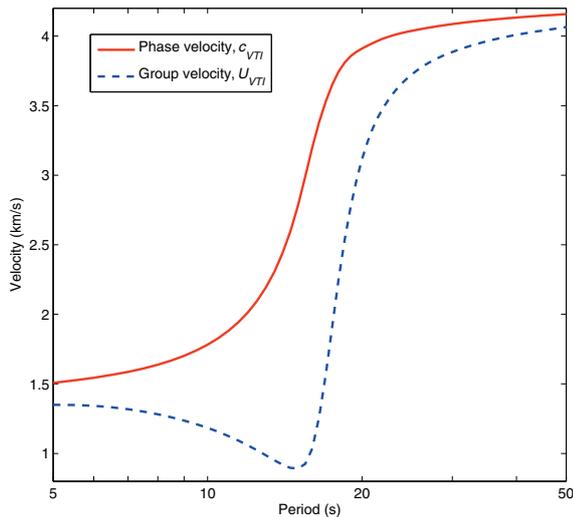


Figure 2. Phase and group velocities of fundamental mode Rayleigh waves for the isotropic crust over anisotropic upper mantle of the Pacific region of age 110+ Myr (Table 1a).

8.1 Oceanic structure

Let us consider a crust and upper mantle structure of the Pacific region of age 110+ Myr [Nishimura and Forsyth, 1989]. We consider 5 layers for isotropic crust over 5 layers of VTI subcrust (Table 1a). Figure 2 shows the phase (c_{VTI}) and group (U_{VTI}) velocities of fundamental mode Rayleigh waves. During computation, complex r_j appears in subcrustal layers for periods up to 18 s. The Table 1b gives the numerical values of c_{VTI} , U_{VTI} and ϵ_{VTI} , which can be used to test other codes.

8.2 Continental structure

As a continental structure, let us consider the VTI model obtained by Huang et al. [2010] for SW China using Love and Rayleigh waves up to the period 40 s.

Layer No.	Thickness (km)	α_H (km/s)	β_V (km/s)	ϕ	η	ρ gm/cm ³
1	6.1	1.5000	0.0000	1.0000	1.0000	1.03
2	0.3	2.0000	0.5000	1.0000	1.0000	1.50
3	1.0	5.6000	3.3000	1.0000	1.0000	2.60
4	2.0	6.4000	3.7000	1.0000	1.0000	2.80
5	3.0	7.2000	4.1000	1.0000	1.0000	3.00
6	2.6	8.1780	4.6357	0.9868	1.0000	3.3995
7	5.0	8.1747	4.6232	0.9786	1.0024	3.3924
8	10.0	8.1699	4.6051	0.9666	1.0094	3.3819
9	10.0	8.1677	4.5947	0.9620	1.0176	3.3669
10	∞	8.1618	4.6222	0.9712	1.0169	3.3530

Table 1a. Structure of the Pacific region with age 110+ Myr [based on Nishimura and Forsyth, 1989].

Period (s)	c_{VTI} (km/s)	U_{VTI} (km/s)	ϵ_{VTI}
5.000	1.5078	1.3497	0.9109
10.000	1.7819	1.1842	0.1457
15.000	2.7796	0.8923	-0.2401
20.000	3.9105	3.1159	-0.6831
25.000	4.0378	3.7239	-0.7429
30.000	4.0863	3.8858	-0.7505
35.000	4.1143	3.9625	-0.7483
40.000	4.1333	4.0092	-0.7438
50.000	4.1578	4.0654	-0.7346

Table 1b. Numerical values of phase velocity (c_{VTI}), group velocity (U_{VTI}) and surface ellipticity (ϵ_{VTI}) of fundamental mode of Rayleigh wave at a few sample periods for the oceanic VTI model in Table 1a.

Layer No.	Thickness (km)	α_H (km/s)	β_H (km/s)	β_V (km/s)	ϕ	η	ρ gm/cm ³
1	16.0	5.63	3.25	3.20	0.9695	1.02	2.5
2	16.0	6.15	3.55	3.34	0.8852	1.02	2.7
3	17.0	6.70	3.87	3.63	0.8798	1.02	3.0
4	∞	7.70	4.39	4.28	0.9505	1.01	3.3

Table 2a. Example of a VTI continental structure [based on Huang et al. 2010].

Period (s)	c_{VTI} (km/s)	U_{VTI} (km/s)	ϵ_{VTI}
5.000	2.9389	2.9264	-0.6765
10.000	2.9716	2.8784	-0.6719
15.000	3.0323	2.8244	-0.6691
20.000	3.1229	2.7448	-0.6664
25.000	3.2457	2.6857	-0.6650
30.000	3.3844	2.7254	-0.6715
35.000	3.5085	2.8842	-0.6914
40.000	3.5996	3.0878	-0.7219
50.000	3.7011	3.3980	-0.7870

Table 2b. Numerical values of phase velocity (c_{VTI}), group velocity (U_{VTI}) and surface ellipticity (ϵ_{VTI}) of fundamental mode of Rayleigh wave at a few sample periods for the continental VTI model in Table 2a.

For the model, they evaluated only β_H and β_V considering $\xi = \phi = \eta = 1$. We have adopted this model and

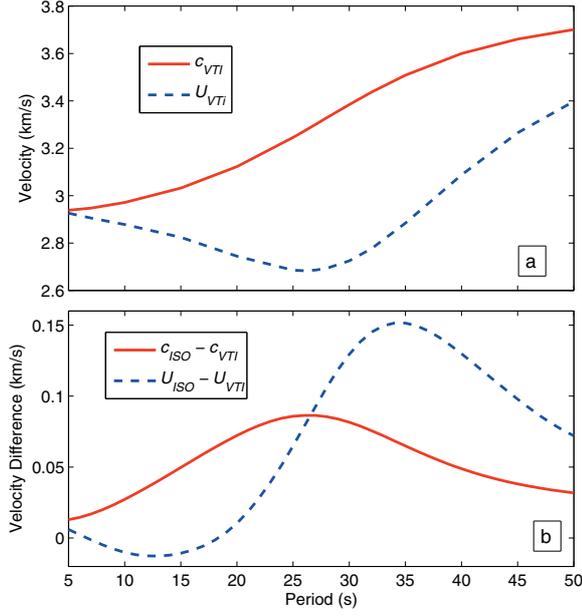


Figure 3. (a) Phase velocity (c_{VTI}) and group velocity (U_{VTI}) of fundamental mode Rayleigh waves for the continental VTI structure in Table 2a. (b) c_{ISO} and U_{ISO} are respectively phase and group velocities of fundamental mode Rayleigh waves for an isotropic structure with $\phi = \eta = 1$ in all the layers in Table 2a. The differences ($c_{ISO} - c_{VTI}$) and ($U_{ISO} - U_{VTI}$) are shown.

considered $\alpha_H = 1.732\beta_H$ and $\alpha_V = 1.732\beta_V$; thus $\phi = [\alpha_V/\alpha_H]^2$; β_H or ξ has no effect on Rayleigh waves. A small value of η has been considered in the adopted model (Table 2a). For this model Figure 3a shows the phase velocities (c_{VTI}) and group velocities (U_{VTI}) of fundamental mode Rayleigh waves. During computation, complex r_1 and r_2 appear at least in one of the lower layers up to period 39 s. An isotropic model is considered with $\phi = \eta = 1$ in Table 2a and the corresponding phase velocities (c_{ISO}) and group velocities (U_{ISO}) are obtained. The difference ($c_{ISO} - c_{VTI}$) is as large as 0.086 s around the period 26 s; further, the difference ($U_{ISO} - U_{VTI}$) is as large as 0.15 around the period 35 s (Figure 3b). Surface ellipticity (ϵ) is shown in Figure 4. The table 2b gives the numerical values of c_{VTI} , U_{VTI} and ϵ_{VTI} .

As indicated in the Introduction, there are a few studies where observed surface wave velocities are initially inverted to get an isotropic model for which theoretical dispersion curves for Love and Rayleigh waves are plotted along with the observed velocity data. In a region of positive S-wave anisotropy ($\beta_H > \beta_V$), these theoretical velocities for Love waves are lesser than the observed data, while those for Rayleigh waves are more than the data on an average. The fit of theoretical curves to the observed data is improved by considering S-wave velocity as β_H with $\xi = 1$ for Love waves and S-wave velocity as β_V with $\phi = \eta = 1$ for Rayleigh waves with $\beta_H > \beta_V$. In such a case, we can use the code of a

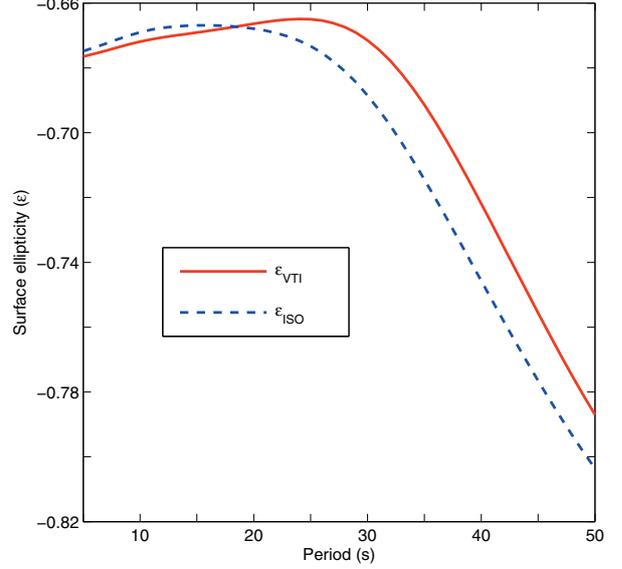


Figure 4. Surface ellipticity (ϵ_{VTI}) of fundamental mode Rayleigh waves for the continental VTI structure in Table 2a. Surface ellipticity (ϵ_{ISO}) is also shown for an isotropic structure with $\phi = \eta = 1$ in all the layers in Table 2a.

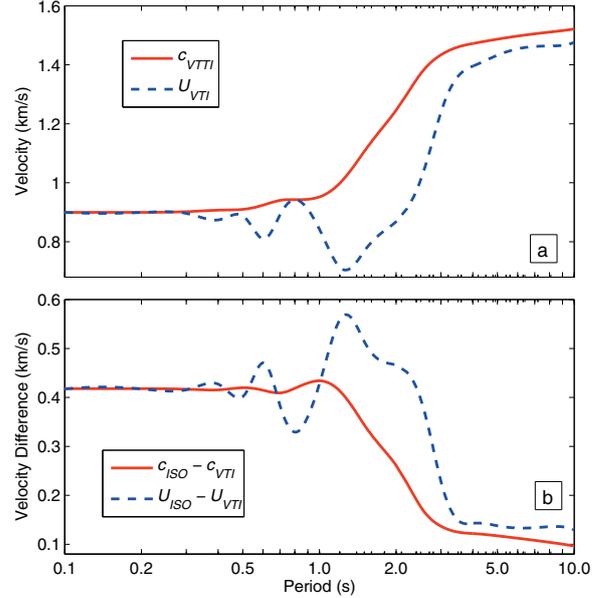


Figure 5. (a) Phase velocity (c_{VTI}) and group velocity (U_{VTI}) of fundamental mode Rayleigh waves for the subsurface VTI structure in Table 3. (b) c_{ISO} and U_{ISO} are respectively phase and group velocities of fundamental mode Rayleigh waves for the isotropic structure with $\phi = \eta = 1$ in all the layers in Table 3. The differences ($c_{ISO} - c_{VTI}$) and ($U_{ISO} - U_{VTI}$) are shown.

layered isotropic half space. However, Figure 3b shows that consideration of $\phi \neq 1$ and $\eta \neq 1$ significantly changes the Rayleigh wave velocities. Thus reducing Rayleigh wave theoretical velocities only by β_V may give lower β_V than actual and this overestimates S-wave anisotropy ξ . This was also explained by Bhattacharya [2015] through sensitivity of Love and Rayleigh wave velocities to VTI medium parameters. A similar erro-

neous estimate of ξ may be obtained in a region with negative S-wave anisotropy ($\beta_H < \beta_V$) for considering $\phi = \eta = 1$.

8.3 Subsurface structure

Surface waves have been used for near surface studies [Socco et al. 2010, Dal Moro 2014]. Consideration of VTI gives better subsurface structure in exploration seismology. In a VTI subsurface, Levin [1979] considered reflections from interfaces. Thomsen [1986] derived relations between NMO (normal move out) velocities and anisotropic parameters. These studies encouraged to obtain subsurface structure through anisotropic travel time inversion and NMO velocity analysis [Tsvankin and Thomsen 1995, Alkalifah and Tsvankin 1995]. In such studies the five stiff elastic coefficients of VTI medium are replaced by α_0, β_0 and the Thomsen parameters γ, ε_T and δ given by

$$\alpha_0 = \sqrt{c_{33}/\rho} = \alpha_V, \beta_0 = \sqrt{c_{44}/\rho} = \beta_V, \quad (49)$$

$$\gamma = \frac{c_{66} - c_{44}}{2c_{44}}, \varepsilon_T = \frac{c_{11} - c_{33}}{2c_{33}}$$

and

$$\delta = \frac{(c_{13} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})} \quad (50)$$

These equations can be used to convert Thomsen parameters to the present representation of VTI parameters. From equation (49) we obtain ϕ from

$$1 + 2\varepsilon_T = \frac{c_{11}}{c_{33}} = \frac{\alpha_H^2}{\alpha_V^2} = \frac{1}{\phi}$$

Thus $\alpha_H = \alpha_V/\sqrt{\phi}$. Further (50) gives

$$\sqrt{[(\alpha_0^2 - \beta_0^2)\{(2\delta + 1)\alpha_0^2 - \beta_0^2\}] - \beta_0^2} = c_{13}/\rho = \frac{F}{\rho} = V_F^2, \text{ say } (51)$$

Hence we obtain, $\eta = V_F^2/(\alpha_H^2 - 2\beta_V^2)$, where V_F^2 is given by left hand side of (51).

Xiao et al. [2004] compared four reflection-travel-time inversion methods to get VTI subsurface structure on synthetic data based on a 4-layered structure shown in Table 3, where the VTI parameters of Xiao et al. [2004] has been converted to the present representation; approximate density values have been considered. Figure 5a shows the phase velocities (c_{VTI}) and group velocities (U_{VTI}) of fundamental mode Rayleigh waves for this model (Table 3). During computation complex r_1 and r_2 appear at least in one of the lower layers even up to period 10 s. We obtain phase velocities (c_{ISO}) and group velocities (U_{ISO}) for an isotropic model by con-

sidering $\phi = \eta = 1$ in Table 3. The difference ($c_{ISO} - c_{VTI}$) is as large as 0.43 s around the period 1 s (Figure 5b). Even at a very low period of 0.1 s, this difference is 0.42 s. The difference ($U_{ISO} - U_{VTI}$) is as large as 0.57 around the period 1.25 s. When period tends to zero, the phase velocity tends to that of the top layer as a half space. The phase velocity top VTI layer as a half space is 0.899 km/s using the dispersion equation of Stoneley [1963] and Vinh and Ogden [2004]; velocity of such a half space decreases as S_3^2 decreases. While considering the top layer as an isotropic half space ($\phi = \eta = 1$), we have phase velocity as $0.9194 * \beta_V = 0.9194 * 1.4 \text{ km/s} = 1.287 \text{ km/s}$ [Ewing et al. 1957]. Thus the difference between Rayleigh wave velocities of the two half spaces are 0.388 km/s. Grechka et al. [2002] noted that the reflection data cannot be uniquely inverted for VTI parameters without additional information. The large velocity difference in Figure 5b shows that short period phase and group velocities can be used to improve the result of anisotropy in subsurface obtained through anisotropic travel time inversion and NMO velocity analysis. The sensitivities of surface wave velocities to the medium parameters are significant down to about half the wavelength of the wave [Bhattacharya 2015]; the short period range may be considered accordingly.

Layer No.	Thickness (km)	α_H (km/s)	β_V (km/s)	ϕ	η	ρ gm/cm ³
1	0.5	3.3130	1.400	0.7143	1.1882	2.2
2	0.5	3.4206	1.500	0.7692	1.1523	2.3
3	0.5	3.5055	1.600	0.8333	1.0780	2.4
4	∞	3.7695	1.750	0.8621	1.0395	2.5

Table 3. Example of a VTI subsurface [Xiao et al. 2004].

9. Conclusions

- In a VTI medium, when $S_3^2 \geq 0$, the decay factor r_j ($j = 1, 2$) is real or imaginary as in a isotropic medium.
- In a VTI medium, when $S_3^2 < 0$, the decay factor r_j becomes complex ($j = 1, 2$). This occurrence depends on the medium parameters and c. In the $\phi - \eta$ plane, the complex region of r_j is bounded by two nearly parallel lines for given c/β_V and α_H/β_V (Figure 1). The area of this region decreases with increase of c/β_V , or α_H/β_V . We find that $S_3^2 < 0$ only if $c < \beta_V$.
- We obtain a layer matrix and a delta layer matrix with real elements when r_j is real or imaginary. When r_j is complex, r_1 and r_2 are complex conjugates; using this property, we get these matrices with real elements.
- Using T-H method with reduced delta matrix,

we obtain Rayleigh dispersion equation (47) with real terms in a layered VTI half space.

- (e) We evaluate Rayleigh wave phase and group velocities for oceanic and continental VTI models through dispersion equation (47) (Figures 2, 3 and 5). Surface ellipticity has been evaluated from equation (48) (Figure 4).
- (f) The dispersion equation (47) contains real terms as in isotropic case. Thus for inversion of observed surface waves to get VTI structure, it may not be necessary to simplify the structure assuming $\phi = \eta = 1$ to compute dispersion curves of Rayleigh waves with isotropic codes as has been done in a few studies.
- (g) Strong VTI is seen in subsurface particularly with organic rich shales. Such subsurface shows significant difference between dispersion curves of Rayleigh waves for VTI and isotropic subsurface (Figure 5). Thus short period surface waves dispersion data can be used to improve the VTI parameters of the subsurface in conjunction with reflection studies in exploration seismology.

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