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An in-depth analysis on the Quasi-Longitudinal approximations applied to ionospheric ray-tracing, oblique and vertical sounding, and absorption

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Abstract

For the phase refraction index of high frequency (HF) waves in the ionospheric medium exists a well-established theory. However, under the Quasi-Longitudinal (QL) conditions, scientific literature presents various formulas that are not equivalent and that, in some cases, give rise to wrong results. In the present study, further consequences of Booker's rule are discussed, illustrating the validity ranges of the above-mentioned approximate formulas; and the different regimes for applying such QL formulas are described, along with the consequences in simulating the ionospheric HF ray-tracing, oblique and vertical sounding, and absorption.

Keywords: Appleton-Hartree's formula; Booker's rule; "Strong" and "weak" Quasi-Longitudinal (QL) conditions; Walker's approximation; Ionospheric HF ray-tracing, oblique and vertical sounding, and absorption.

1. Introduction

In the scientific literature, approximations to the Appleton-Hartree's formula have been considered by several authors [Booker, 1935; Rydbeck, 1940; Westfold, 1951; Ratcliffe, 1959; Titheridge, 1959]. In most cases, however, the range of validity of the expressions used is difficult to assess. Although, with the advent of electronic computers, the necessity for using approximations in numerical analysis has decreased, anyway they are widely used to facilitate theoretical discussions about ionospheric absorption if the propagation is considered as occurring in straight line. Davies and King [1961] considered in their paper the applicability of some approximate formulas for the case of the Earth's dipole magnetic field and for frequencies normally used in ionospheric sounding at vertical incidence.

For purposes of discussion, it has often been considered convenient to use approximations of the Appleton-Hartree's formula that involve the relative magnitudes of the terms under the square root. These approximations are usually referred to as Quasi-Transverse (QT) and Quasi-Longitudinal (QL). Besides, these approximations have extensively been discussed including collisions by Ratcliffe [1959], Kelso [1964], Budden [1988], and Davies [1990]. However, the importance of taking into account Booker's rule is not adequately emphasized. Care should be exercised in the use of these formulas because the approximations may depend on how the values of phase refractive index approach the transverse and longitudinal values. They should not be used near reflection levels with vertical propagation [Davies, 1990].

Scotto and Settimi [2014] proposed new outcomes for ionospheric absorption starting from the Appleton-Hartree's formula, in its complete form. The range of applicability was discussed for the approximate formulae, which are usually employed in the calculation of non-deviative absorption coefficient. These results were achieved by performing a more refined approximation that is valid under QL propagation conditions. The more refined QL approximation and the usually employed non-deviative absorption were compared with that derived from a complete formulation. Their expressions, nothing complicated, can usefully be implemented in a software program running on modern computers. Moreover, the importance of considering Booker's rule was highlighted. A radio link of ground range D = 1000 km was also simulated using a simplified ray-tracing, without magnetic field, for a typical daytime ionosphere. Finally, some estimations of the ionospheric absorption integrated along the radio link considered were provided for different frequencies.

Concluding, for the phase refraction index of high frequency (HF) waves in the ionospheric medium exists a well-established theory. However, under the Quasi-Longitudinal (QL) conditions, scientific literature presents various formulas that are not equivalent and that, in some cases, give rise to wrong results. In the present study, further consequences of Booker's rule are discussed, illustrating the validity ranges of the above-mentioned approximate formulas; and the different regimes for applying such QL formulas are described, along with the consequences in simulating the ionospheric HF ray-tracing, oblique and vertical sounding, and absorption.

The study is organized as follows. Following this Section 1, as an introduction, Section 2 defines the Appleton-Hartree's formula, and Booker's rule; Section 3 compares the so-called "strong" or "weak" QL conditions, and analysing Walker's [1961] approximation; Section 4 is structured into ionospheric HF ray-tracing, oblique and vertical sounding, and absorption simulations. Section 5 reports the results and analysis. Moreover, Section 6 draws up the discussion and conclusions. Finally, in Section 7, the Appendix will provide an outline of the somewhat lengthy calculations needed to demonstrate, for the first time to the best of author's knowledge, a so-called "Y–Walker's" QL condition, which leads to an implicit relation of dispersion for the magneto-plasma linking its ionospheric parameters, never discussed in scientific literature.

2. Appleton-Hartree's formula, and Booker's rule

The complex phase refractive index for radio waves in the ionosphere, considering the effects of both the geomagnetic induction field and electron-neutral particle collisions, is given by the Appleton-Hartree's formula [Budden, 1988]:

$$n_{\mathrm{ord,ext}}^2 = \left(\mu_{\mathrm{ord,ext}} - i\chi_{\mathrm{ord,ext}}\right)^2 = 1 - \frac{X}{1 - iZ - \dfrac{Y_{\mathrm{T}}^2}{2\left(1 - X - iZ\right)} \pm \sqrt{\dfrac{Y_{\mathrm{T}}^4}{4\left(1 - X - iZ\right)^2} + Y_{\mathrm{L}}^2}}, \tag{1}$$

where the real part μ is the refractive index, the imaginary part χ is proportional to the absorption coefficient, and:

$X = \omega_{\mathrm{p}}{}^2/\omega^2$ (being ω the angular frequency of the radio wave, $\omega_{\mathrm{p}} = \sqrt{Ne^2/m\varepsilon_0}$ the plasma frequency, N the electron density, m the electron mass, e the electron charge, and θ_0 the dielectric constant of vacuum);

$Y_{\mathrm{T}} = Y{\cdot}\sin(\theta)$, $Y_{\mathrm{L}} = Y{\cdot}\cos(\theta)$ (being θ the angle between the wave vector and the direction of geomagnetic induction field), and $Y = \omega_{\mathrm{B}}/\omega$ (being $\omega_{\mathrm{B}} = B|e|/m$ the angular gyro-frequency, and B the amplitude of geomagnetic induction field);

$Z = \nu/\omega$ (being ν the collision frequency).

For the known birefringence of ionospheric plasma, this relationship allows to derive two refractive indices, for the ordinary ray n_{ord} and the extraordinary ray n_{ext}, where the refractive indices $n_{\mathrm{ord,ext}}$ are complex quantities (being $n_{\mathrm{ord}} = \mu_{\mathrm{ord}} - i\cdot\chi_{\mathrm{ord}}$ and $n_{\mathrm{ext}} = \mu_{\mathrm{ext}} - i\cdot\chi_{\mathrm{ext}}$, with obvious meaning of symbols). The two refractive indices are obtained from Eq. (1) through the choice of positive or negative signs, which must be decided applying the so-called Booker's [1935] rule. Once the Booker's critical frequency $\omega_{\mathrm{c}} = (\omega_{\mathrm{B}}/2)\cdot\sin^2(\theta)/\cos(\theta)$ is defined, whereas the angular gyro-frequency is equal to $\omega_{\mathrm{B}} = 2\pi f_{\mathrm{B}}$, being $f_{\mathrm{B}} = 1.2$ MHz at medium latitudes and ionospheric altitudes, Booker's rule states that, to achieve continuity of μ_{ord} (μ_{ext}) and χ_{ord} (χ_{ext}), if $\omega_{\mathrm{c}}/\nu > 1$, the positive (negative) sign in Eq. (1) must be adopted both for $X < 1$ and for $X > 1$; instead, if $\omega_{\mathrm{c}}/\nu < 1$, the positive (negative) sign for $X < 1$ and negative (positive) for $X > 1$ must be adopted.

3. "Strong" and "Weak" Quasi-Longitudinal (QL) conditions, and Walker's approximation

In order to simplify the above Eq. (1), the QL and QT approximations [Budden, 1988], first discussed by Booker [1935], are often used.

In this discussion, it is considered that the "strong" QL condition holds when the term $\frac{Y_{\mathrm{T}}^2}{2\left(1-X-iZ\right)}$ can be neglected, i.e.:

$$\frac{Y_{\mathrm{T}}^2}{2Y_{\mathrm{L}}} \ll \sqrt{\left(1-X\right)^2+Z^2}. \tag{2}$$

The QL approximation (appearing Y_{L}) is generally quoted as:

$$n^2_{\mathrm{ord,ext(QL)}} \simeq 1-\frac{X}{1-iZ \pm Y_L} \tag{3}$$

Eq. (3) means that the wave behaves very much as if it is being propagated along the direction of the geomagnetic induction field. Eq. (3) is valid in first-order approximation, as can be easily demonstrated by considering the denominator of the second term of Eq. (1): $1-iZ-\frac{Y_{\mathrm{T}}^2}{2\left(1-X-iZ\right)} \pm \sqrt{\frac{Y_{\mathrm{T}}^4}{4\left(1-X-iZ\right)^2}+Y_{\mathrm{L}}^2}$. When the inequality (2) holds, this becomes: $1 - iZ \pm Y_{\mathrm{L}}$, and $\frac{Y_{\mathrm{T}}^2}{2\left(1-X-iZ\right)}$ is necessarily negligible compared with Y_{L}. In fact, if we assume the usual numerical condition that the inequality (2) is satisfied if the larger quantity is nine times the smaller, then $\left|\frac{Y_{\mathrm{T}}^2}{2\left(1-X-iZ\right)}\right|$ may be much smaller than Y_{L}.

The Longitudinal (L) approximation (appearing Y) is generally quoted as:

$$n^2_{\mathrm{ord,ext(L)}} \simeq 1-\frac{X}{1-iZ \pm Y}. \tag{4}$$

Eq. (3), concerning the QL approximation (by Y_{L}), provides, under a condition of reflection $n_{\mathrm{ord,ext(QL)}} = 0$, some miscalculated critical frequencies of penetration for both the ordinary and extraordinary rays: $X \approx 1 \pm Y_{\mathrm{L}}$ (neglecting the electron-neutral collision effects in an ionospheric magneto-plasma, i.e. $Z \ll 1$). Moreover, Eq. (4), concerning the L approximation (by Y), yields a correct penetration critical frequency just for the extraordinary ray: $X \approx 1 - Y$.

The "weak" QL condition holds when:

$$\frac{Y_{\mathrm{T}}^4}{4Y_{\mathrm{L}}^2} \ll \left(1-X\right)^2+Z^2. \tag{5}$$

Walker's [1961] approximation is quoted as:

$$n^2_{\mathrm{ord,ext(WALKER)}} \cong 1 - \frac{X}{1 - iZ - \dfrac{Y_T^2}{2\left(1 - X - iZ\right)} \pm Y_L}. \qquad (6)$$

Eq. (6) means that the wave does not behave as if it is being propagated along the direction of the geomagnetic induction field. Eq. (6) is valid in second-order approximation, when inequality (5) holds, and beyond the limits of inequality (2). Indeed, Eq. (2) may be derived by square-rooting Eq. (5), such that the "strong" QL condition (2), under which can be applied the QL approximation (Y_L) (3), holds in a band of the plasma frequency parameter X shorter than the corresponding band of the "weak" QL condition (5), under which Walker's approximation (6) can be applied. Finally, in a collision-less magneto-plasma $Z \ll 1$, the "weak" QL condition (5) is simplified as $\frac{Y_T^4}{4Y^2}\frac{1}{\left(1-X\right)^2} \ll 1$, and, under the reflection condition $n_{\mathrm{ord,ext(WALKER)}} = 0$, Walker's approximation (6) is reduced to $\left(\frac{1-X}{Y}\right)^2 \simeq 1 - \frac{Y_T^4}{4Y^2\left(1-X\right)^2} \approx 1$, so that, once again, the critical frequency just for the extraordinary ray is correctly deduced: $X \approx 1 - Y$.

Let us compare the Appleton-Hartree's formula and Booker's rule, under the "strong" or "weak" QL conditions, against the QL (appearing Y_L), the L (appearing Y) or Walker's approximations.

In Figure 1, the real part of phase refractive index μ is plotted as a function of the plasma frequency parameter X, when the ordinary and extraordinary rays are modelled: by Appleton-Hartree's general formulation [Eq. (1)]; moreover, holding the "strong" Quasi-Longitudinal (QL) condition [Eq. (2)], by the QL approximation (appearing

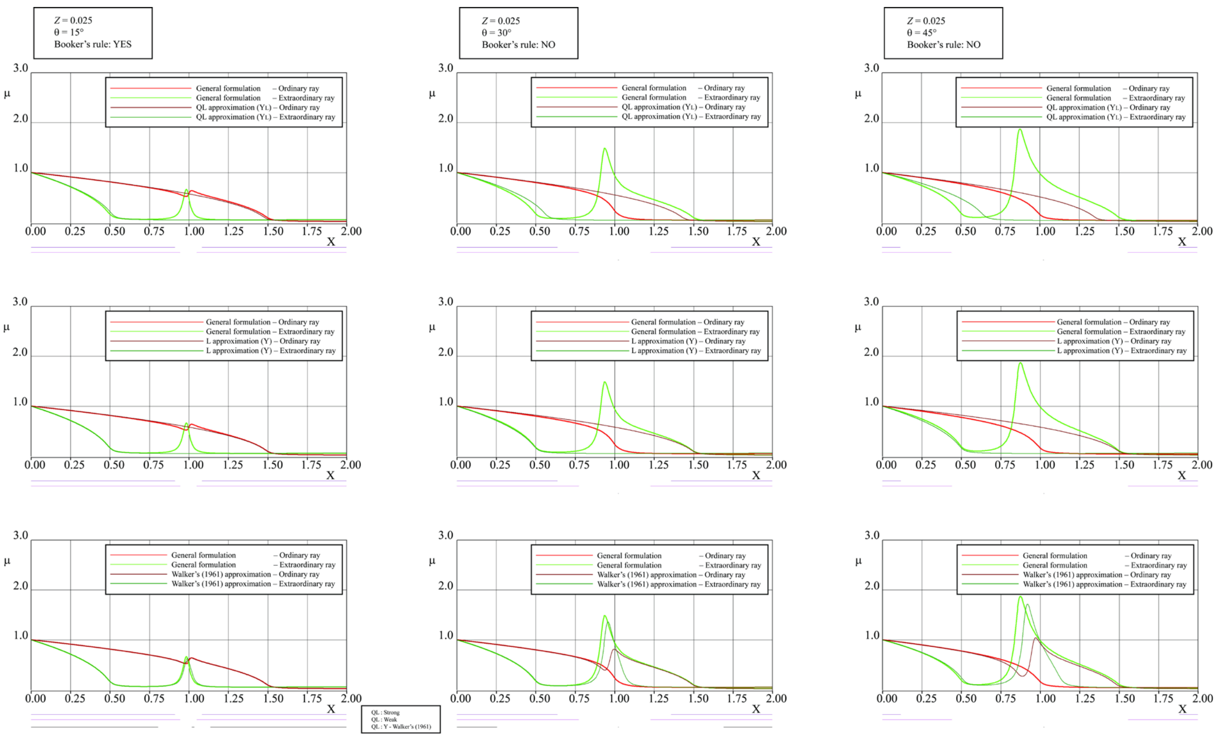


Figure 1. The real parts of ordinary and extraordinary phase refractive index μ_{ord} and μ_{ext} are plotted as a function of the plasma frequency parameter X. The "strong", "weak" and "Y–Walker's" QL conditions [see Appendix, Eq. (A.2)] are shown for the fixed collision frequency parameter ($Z = 0.025$), and increasing the geomagnetic field angle ($\theta = 15°, 30°, 45°$), in account of Booker's rule.

Y_L) [Eq. (3)] and the Longitudinal (L) approximation (appearing Y) [Eq. (4)]; and, finally, holding the "weak" QL condition [Eq. (5)], by Walker's approximation [Eq. (6)]. The "strong", "weak" and "Y–Walker's" QL conditions [see Appendix, Eq. (A.2)] are shown for the fixed collision frequency parameter ($Z = 0.025$), and increasing the geomagnetic field angle ($\theta = 15°, 30°, 45°$), in account of Booker's rule (It is reported "YES" when Booker's rule imposes the sign change at $X = 0$, "NO" otherwise).

As comments of Figure 1, if Booker's rule imposes the sign change, then Appleton-Hartree's general formulation is well approached by the QL (Y_L), the L (Y) and Walker's approximations, on almost the whole range of plasma frequency parameter X, wherein is holding the "strong" or "weak" QL condition. Moreover, the QL approximation (Y_L) is not reliable when $X \approx 1 - Y$ and $X \approx 1$, while the L approximation (Y) just when $X \approx 1$. Finally, Walker's approximation becomes reliable even around $X \approx 1$. Instead, if Booker's rule does not impose a sign change, then Appleton-Hartree's general formulation is no way approached by the QL (Y_L), the L (Y) and Walker's approximations, on the range of plasma frequency parameter X such that $1 < X \leq 1 + Y$, wherein is not holding the "strong" or "weak" QL condition. Moreover, the QL approximation (Y_L) is more accurate than the L approximation (Y) as a model just of the ordinary ray, when holding its non deviative absorption condition $X \rightarrow 0$; vice-versa, the L approximation (Y) is more accurate than the QL approximation (Y_L) as a model just of the extraordinary ray, when holding its deviative absorption condition $X = 1 - Y$. Finally, Walker's approximation becomes less reliable just straddling the step point $X \approx 1$.

If Booker's rule does not impose a sign change, raising the geomagnetic field angle θ, then Appleton-Hartree's general formulation is no well approached by the QL (Y_L), the L (Y) and Walker's approximations, even on the range of plasma frequency parameter X such that $X \leq 1$, wherein is holding the "strong" or "weak" QL condition. Moreover, raising the geomagnetic field angle θ, though, in the range $X \leq 1$, the ordinary ray is worse fitted by all the three approximations, anyway, in the range $1 < X \leq 1 + Y$, the ordinary ray is better, or equally, and either worse fitted, respectively by the QL (Y_L), or the L (Y), and either Walker's approximations. Finally, independently from holding the "strong" or "weak" QL condition, in the range $X \leq 1$, the extraordinary ray is worse fitted by the QL approximation (Y_L), and better, though no more well, fitted by the L (Y) and Walker's approximations. Instead, in the range $1 < X \leq 1 + Y$, the extraordinary ray is worse fitted by all the three approximations.

The "strong" QL condition (2), under which can be applied the QL approximation (Y_L) (3), holds in a band of the plasma frequency parameter X shorter than the corresponding band of the "weak" QL condition (5), under which can be applied Walker's approximation (6). Moreover, if the geomagnetic field angle θ is raised, then both the "strong" and "weak" QL bands are shortened, thus reducing the domain in which the QL (Y_L) and Walker's approximations are superimposable. Finally, the so-called "Y–Walker's" QL condition, under which the L approximation (Y) (4) can be merged with Walker's approximation (6) [see Appendix], holds in a band of the plasma frequency parameter X even shorter than the "strong" and "weak" QL bands. If the geomagnetic field angle θ is raised, then the "Y–Walker's" QL band is shortened more rapidly than the "strong" and "weak" QL bands, until collapsing to an interval of zero length, so that the L (Y) and Walker's approximations are no more superimposable. By the way, Appendix will underline that: the L approximation (appearing Y) [whence Eq. (A.4)], superimposes to Walker's approximation on almost the whole range of plasma frequency parameter X less than around the step point $X \approx 1$, as long as Booker's rule imposes the sign change, consistently with a low geomagnetic field angle θ. Moreover, the L approximation (by Y) is accurate just on a short range of parameter X such that $X \ll 1$, if Booker's rule does not impose a sign change, due to a middle value of θ. Finally, the L approximation (Y) is not at all reliable on any range of X, if Booker's rule does not impose a sign change, when raising to a high θ (see Figure 1).

4. Ionospheric HF ray-tracing, oblique and vertical sounding, and absorption

Ionospheric ray-tracing is a numerical technique used to determine the detailed path of a high frequencies (HF) radio wave propagating in the ionosphere from a transmitting point to a receiving point [Budden, 1988].

Ray-tracing provides a detailed knowledge of radio wave propagation throughout the ionosphere. Examples of main applications are concerning the Over The Horizon (OTH) radar systems, the single station location, the HF direction finding systems, the management of HF radio communications, and the predictions of operating frequencies [Nickisch, 2008].

Accurate ray-tracing is usually performed using techniques that numerally integrate Haselgrove's equations [Haselgrove, 1955; Haselgrove and Haselgrove, 1960]. In the limits of the ray theory, it is possible to approximate the wavelength to zero. The propagation of the wave, i.e. the ray path, is described by: at least six differential equations, where the parameters of both the position and ray direction need to be integrated simultaneously at each point along the ray path; plus other two equations, if additional quantities are required, such as Doppler's frequency shift for instance. The integration provides the coordinates reached by the wave vector and its three components, the group time delay of the wave along the path, and other optional quantities such as geometrical and phase path, polarization, absorption, etc.

Azzarone et al. [2012] described IONORT (IONOspheric Ray Tracing), which is an applicative software tool package for calculating a three-dimensional (3-D) ray tracing of HF radio waves in the ionospheric medium. IONORT uses a 3-D electron density representation of the ionosphere, as well as geomagnetic induction field and electron-neutral particle collision frequency models having validity in the area of interest. An analytical standard Chapman's [1931a; 1931b] modelled ionosphere, useful mainly for testing purpose, completes the whole applicative software tool package.

IONORT-ISP system (IONOspheric Ray-Tracing – IRI-SIRMUP-Profiles) was developed and tested by comparing the recorded oblique ionograms over the radio link between Rome (41.89° N, 12.48° E), Italy, and Chania (35.51° N, 24.02° E), Greece, with IONORT-ISP synthesized oblique ionograms [Pezzopane et al., 2011; Pezzopane et al., 2013; Settimi et al., 2013a]. Based on suitable upgrades concerning both the D-layer and ISP models, a new applicative software tool package, named IONORT-ISP-WC system (WC means with collisions) was developed, and a database of 33 IONORT-ISP-WC synthesized oblique ionograms was calculated for single and multiple ionospheric reflections (1 – 3 hop paths). IONORT-ISP-WC synthesized oblique ionograms were compared with both IONORT-IRI-WC synthesized oblique ionograms, generated by applying IONORT in conjunction with the 3-D International Reference Ionosphere (IRI) electron density profile grid, and the recorded oblique ionograms over the aforementioned radio link [Settimi et al., 2015].

The ionospheric absorption can be discussed assuming a quasi-flat layered ionospheric medium, with small horizontal gradients. A recent complex eikonal model [Settimi et al., 2013b] was applied, useful to calculate the absorption due to the ionospheric D-layer, which can be approximately characterized by a linearized analytical profile of complex refractive index, covering a short range of heights between $h_1 = 50$ km and $h_2 = 90$ km. Moreover, the complex eikonal model for the D-layer was already compared with the analytical Chapman's profile of ionospheric electron density [Settimi et al., 2014a]. Finally, in Settimi et al. [2014b], the simple complex eikonal equations, in QL approximation, for calculating the non-deviative absorption coefficient due to the propagation across the D-layer were encoded into a so-called COMPLEIK (COMPLex EIKonal) subroutine of the IONORT program. As main outcome, the simple COMPLEIK algorithm was compared to the more elaborate semi-empirical ICEPAC formula [Stewart, undated], which refers to various phenomenological parameters such as the critical frequency of E-layer.

In this section, the Appleton-Hartree's formula and Booker's [1935] rule (Sec. 2) will be compared, under the "strong" or "weak" QL conditions, to the QL (appearing Y_L) and L (appearing Y) or Walker's (1961) approximations (Sec. 3), which can be applied to the ionospheric HF ray-tracing, oblique and vertical sounding, and absorption simulations.

4.1 Ionospheric HF ray-tracing

Let us consider a "long" radio link, with a single ionospheric reflection (1 hop), between the transmitter (TX), located in Rome, Italy ($lat_{TX} = 41°53'35''$ N, $lon_{TX} = 12°28'58''$ E) and the receiver (RX), located in Chania, Greece ($lat_{RX} = 35°30'00''$ N, $lon_{RX} = 24°01'00''$ E), as reproduced in Figure 2. International Reference Ionosphere (IRI)-2007 model allows computing the E plasma critical frequency f_oE, on any date and time, relative to the midpoint M between TX and RX, $M = [lat_M = (lat_{TX} + lat_{RX})/2 = 38.70°$ N, $lon_M = (lon_{TX} + lon_{RX})/2 = 18.25°$ E]. IRI-2007 parameter f_{oE} is an input of ICEPAC formula [Stewart, undated], useful to calculate phenomenologically the non-deviative absorption coefficient, in QL approximation. IONORT (IONOspheric Ray-Tracing) program [Azzarone et al., 2012], simulating a three-dimensional (3-D) ray-tracing of high frequency (HF) waves across the ionospheric medium, is used in conjunction with the global and climatological IRI model [Bilitza and Reinisch, 2008] or even the regional and assimilative IRI-SIRMUP-Profiles (ISP) model [Pezzopane et al., 2011; 2013]. IRI (or ISP) model generates a 3-D (even real-time) electron density grid of the ionosphere over Mediterranean area, in order to synthesize oblique ionograms of the long radio link between TX and RX stations, at a some distance away. IONORT-IRI

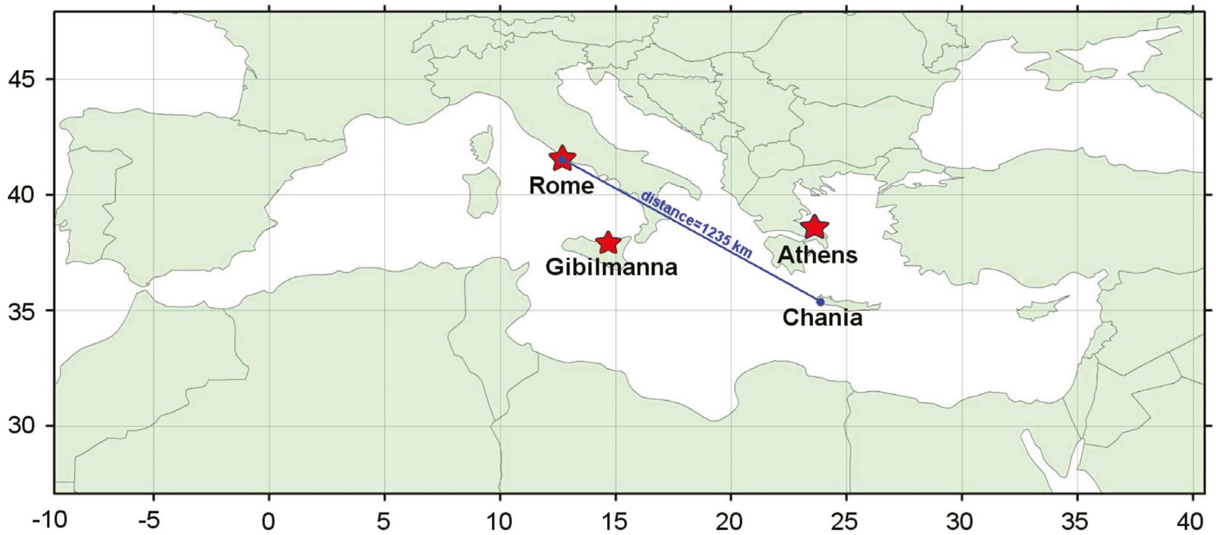


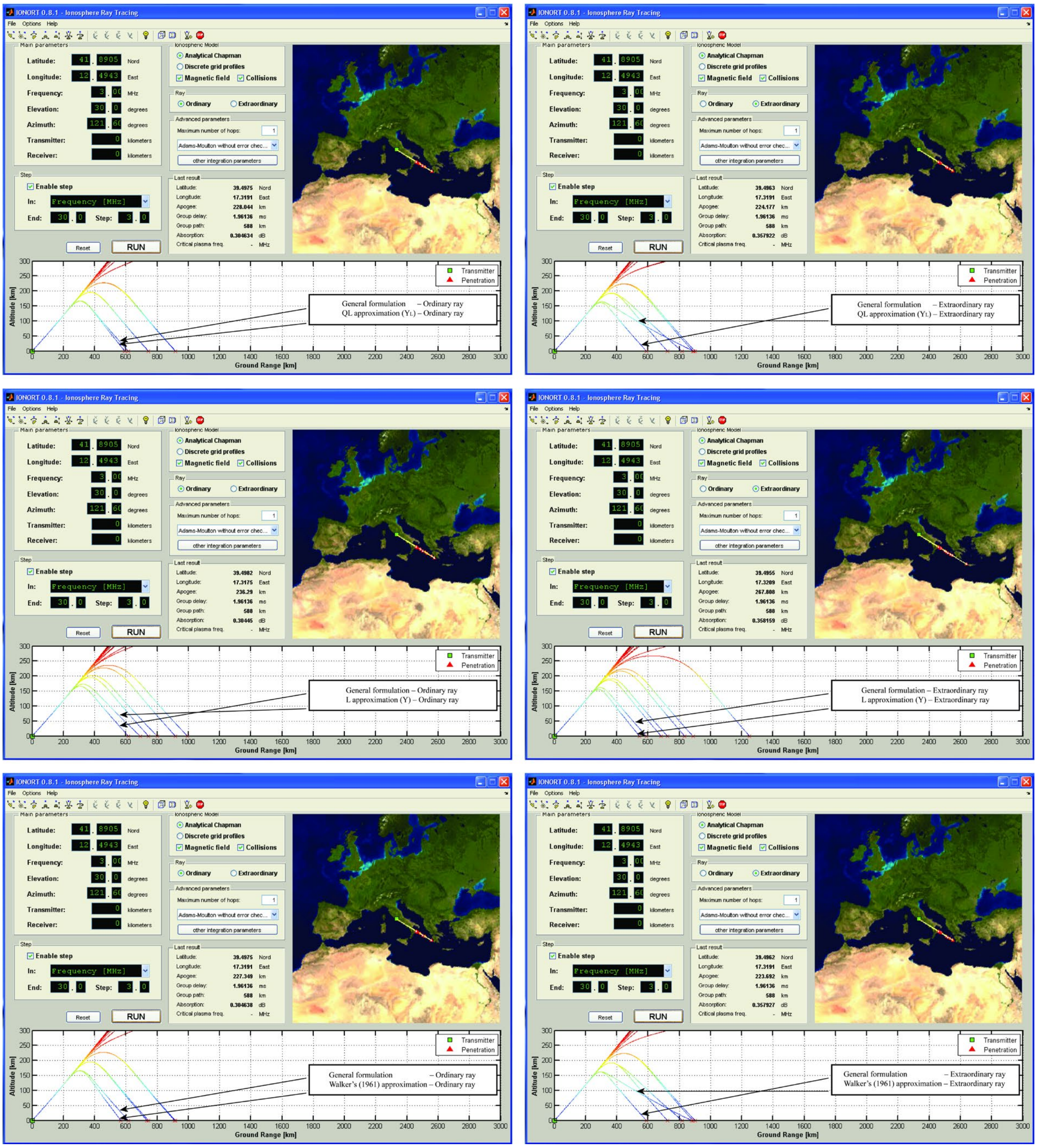
Figure 2. Map of the central Mediterranean region considered in this study. Red stars represent the ionospheric stations used as input for the ISP model. In blue, the radio link between Rome and Chania used to test the effectiveness of the IONORT-ISP system.

and IONORT-ISP systems [Settimi et al., 2013a; 2015] allow computing the height $h_{\max}$ relative to the maximum electron density $N_{\max}$, which is an input of COMPLEIK subroutine (based on the COMPLex EIKonal model) [Settimi et al., 2013b; 2014a; 2014b], useful to calculate theoretically the QL non-deviative absorption coefficient.

Figure 3 presents screen plots of the graphical user interface (GUI) of IONORT program. The two dimensional (2-D) and the latitude-longitude projection of three dimensional (3-D) visualization for the ray paths are shown at the bottom and right of the plots respectively, considering a TX point at Rome (Italy) with azimuth angle of transmission equal to 121.6°, in the direction of a RX point at Chania (Greece). To realize the full potential of IONORT, composite simulations are performed over the central Mediterranean region, based on a suitable Chapman's [Chapman, 1931a; 1931b] electron density, by taking IGRF-12 geomagnetic induction field and Jones-Stephenson's [1975] double exponential profile of electron collision frequency into account. The ordinary and extraordinary rays are modelled by: Appleton-Hartree's general formulation; moreover, the QL approximation (by Y_L) and the L approximation (by Y); and, finally, Walker's [1961] approximation. Two simulations are plotted for: (a) a fixed elevation angle of 30°, with a 3 MHz frequency-step procedure from 3 MHz to 30 MHz (Figure 3.a); and, (b) a fixed frequency of 9 MHz, with a 10° elevation-step procedure from 5° to 85° (Figure 3.b).

As comments of Figure 3.a, the L approximation (Y), when modelling both the ordinary and extraordinary rays, fails to provide the RX point relative to a radio propagation at every frequencies of the whole bandwidth. By the way, under the L approximation (Y), both the ordinary and extraordinary rays are characterized by a critical penetration frequency linearly dependent just on the geomagnetic field amplitude parameter Y, reason why the deviance of RX point could be uniformly distributed throughout the whole frequency band. Note, conforming to Figure 1, that, if Booker's [1935] rule does not impose a sign change, the QL approximation (Y_L) is more accurate than the L approximation (Y) as a model just of the ordinary ray, when holding its non deviative absorption condition $X \to 0$. Even more raising the geomagnetic field angle θ, in the range $1 < X \leq 1 + Y$, the ordinary ray is better fitted by the QL approximation (Y_L) than the L approximation (Y). Moreover, both the QL (Y_L) and Walker's approximations, when modelling both the ordinary and extraordinary rays, fail [succeed] to provide the RX point relative to a radio propagation at lower [higher] frequencies. Note, conforming to Figure 1, that: if Booker's rule does not impose a sign change, then, on the range of plasma frequency parameter X such that $1 < X \leq Y + 1$, the QL approximation (Y_L) is slightly more accurate than Walker's approximation, when modelling just the ordinary ray; and, on the right side of the step point $X \approx 1$, it is almost superimposable to Walker's approximation, when modelling just the extraordinary ray.

As comments of Figure 3.b, the L approximation (Y), when modelling both the ordinary and extraordinary rays, fails to provide the RX point relative to a radio propagation, which is launched at every elevation angles. By the way, under the L approximation (Y), both the ordinary and extraordinary rays are characterized by a critical penea)



tration frequency independent from the geomagnetic field angle θ, reason why the deviance of RX point could be uniformly distributed throughout the whole elevation range. Moreover, the QL approximation (Y_L), when modelling both the ordinary and extraordinary rays, fails (succeeds) to provide the RX point relative to a radio propagation, which is launched at higher (lower) elevation angles. Finally, Walker's approximation, when modelling both the ordinary and extraordinary rays, succeeds to provide the RX point relative to a radio propagation, which is launched at every elevation angles. Deepening the comments on Figure 3.b, by considering either ordinary or extraordinary rays, the L approximation (Y) is not at all reliable, instead Walker's approximation is anyhow more acb)

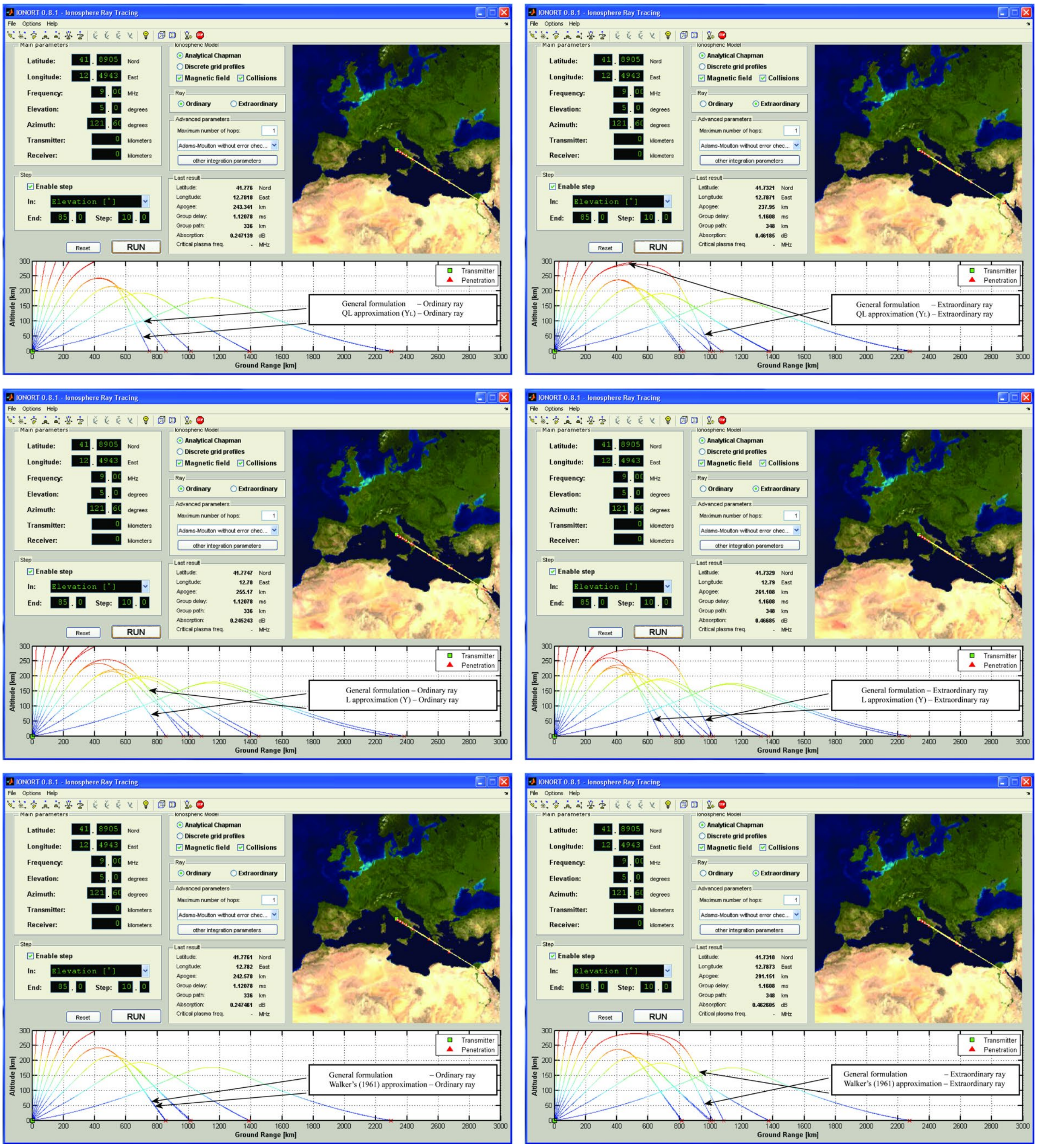


Figure 3. Graphical user interface (GUI) of IONORT program. Two simulations are plotted for: (a) a fixed elevation angle of 30°, with a 3 MHz frequency-step procedure from 3 MHz to 30 MHz; and, (b) a fixed frequency of 9 MHz, with a 10° elevation-step procedure from 5° to 85°.

curate than the QL approximation (Y_L). As a note, the QL (Y_L), L (Y) and Walker's approximations, when modelling just the extraordinary ray, fail generally to provide the RX point relative to a radio propagation, which is launched at the critical elevation angle. Concluding the comments on Figure 3.b, as regards Pedersen's extraordinary ray [Lu, 1996], the QL approximation (Y_L) is not at all reliable, as its Pedersen's ray punches erroneously the ionosphere. Instead, Walker's approximation is much more accurate than the L (Y) approximation, as Pedersen's RX point, expected by Appleton-Hartree's, is far from: the corresponding one, expected by L (Y), more than 300 km; and the corresponding one, expected by Walker's, less than 100 km.

4.2 Oblique sounding and absorption

Figure 4.a reproduces the oblique ionograms recorded along Rome (TX)-Chania (RX) radio link on 8 October 2011 at 6:45 UT, 6 July 2011 at 12:00 UT, and 7 July 2011 at 18:00 UT.

As comments of Figure 4.a, reproducing the recorded oblique ionograms, the TX is based on a VOS-1 chirp ionosonde designed by the Barry Research Corporation, Palo Alto, California, USA [1975], sweeping from 2 to 30 MHz at 100 kHz/s, with an average power of less than 10 W. The RX is a RCS-5B chirp designed by the Barry Research Corporation [1989] (see further details therein Settimi's et al. ref. [2013a]).

Figure 4.b reproduces, with reference to the caption of Figure 4.a, the corresponding oblique ionograms synthesized by IONORT-ISP-WC system. In these oblique ionograms, both the ordinary and extraordinary traces, com-

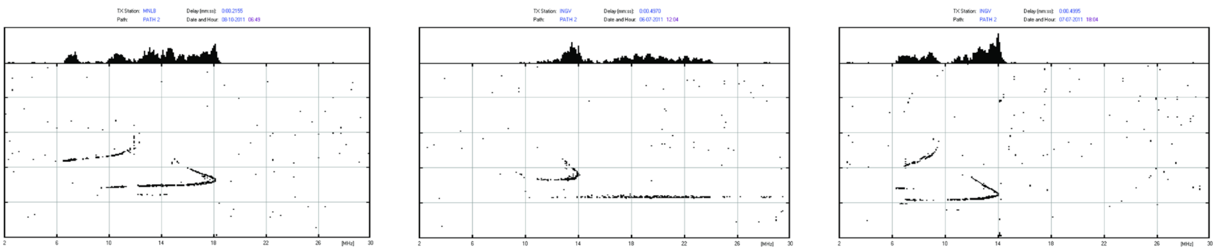


Figure 4.a. Oblique ionograms recorded along Rome (TX)-Chania (RX) radio link on 8 October 2011 at 6:45 UT, 6 July 2011 at 12:00 UT, and 7 July 2011 at 18:00 UT.

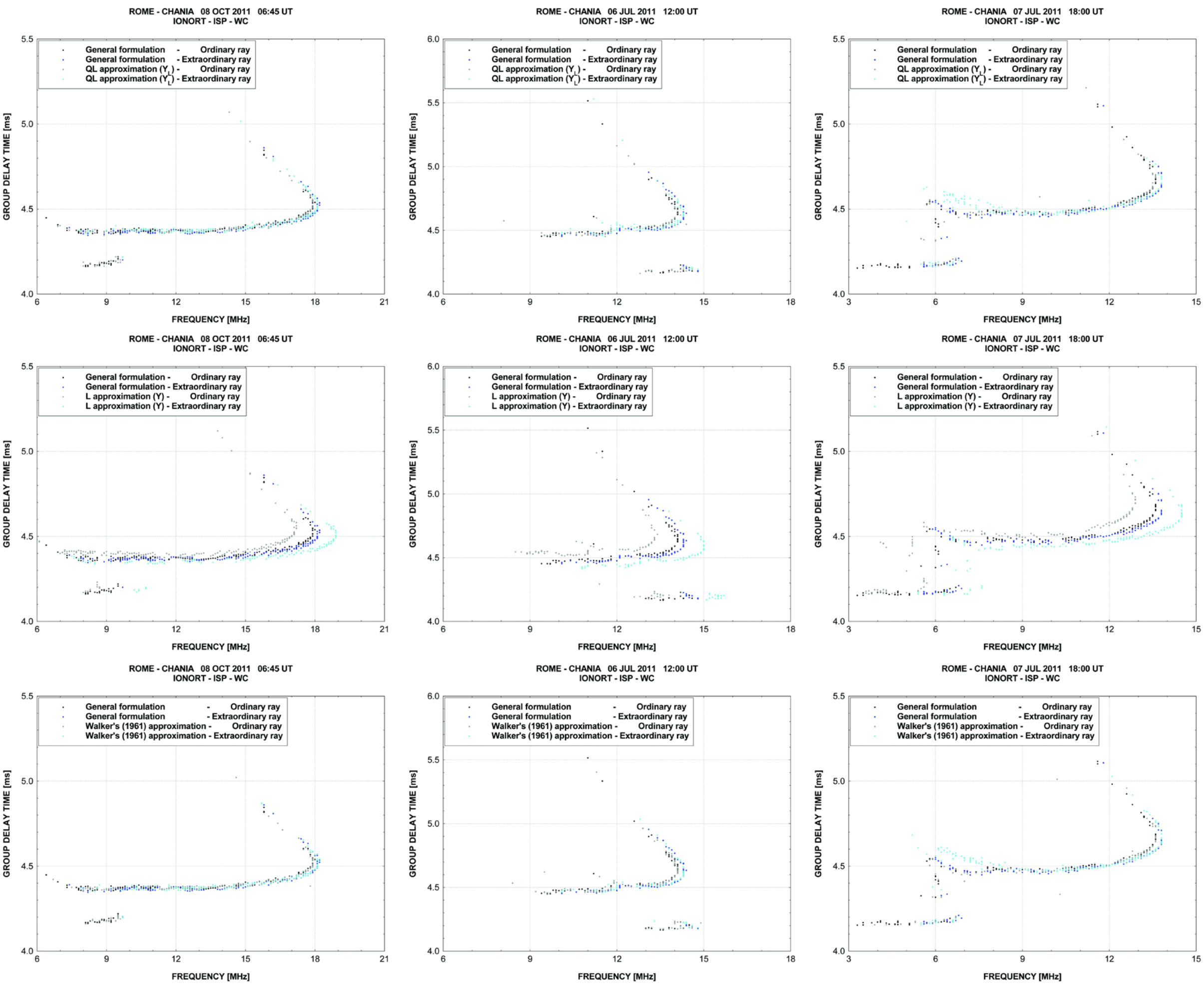


Figure 4.b. Corresponding oblique ionograms synthesized by IONORT-ISP-WC system;.

puted by taking IGRF-12 geomagnetic induction field and Jones-Stephenson's [1975] double exponential profile of electron collision frequency into account, are shown. For one ionospheric reflection (1 hop path), a nested loop cycle was iterated with angular frequency set from 3 MHz to 30 MHz of step 0.1 MHz, moreover the azimuth angle was set from 121° to 122° of step 0.2°, the elevation angle set from 0° to 90° of step 0.2°, and finally the RX range accuracies to 0.1 % , where the RX accuracies are defined as: $(Latitude_k - Latitude_{RX})/Latitude_{RX}$ and $(Longitude_k - Longitude_{RX})/Longitude_{RX}$; being $Latitude_{RX}$, $Longitude_{RX}$ respectively the actual latitude and longitude of RX point; and $Latitude_k$, $Longitude_k$ respectively the latitude and longitude of arrival point as simulated by the k^{th} cycle of ray-tracing.

In Figure 4.b, as a note, the jitter noise is due to some numerical instabilities, which could be offset by adjusting the integration step of the ray path length as a function of the local value for the phase refractive index. However, the feedback would not be so satisfactory for the oblique ionograms synthesized by the discrete IONORT-ISP-WC system, as would be expected using an analytical Chapman's [1931a; 1931b] electron density.

As comments of Figure 4.b, reproducing the synthesized oblique ionograms, the results obtained point out that during daytime, for the lower ionospheric layers, the traces of the synthesized ionograms are cut away at high frequencies because of HF absorption; the IONORT-ISP-WC MUF values are as accurate as the recorded MUF values. The results presented suggest that the IONORT-ISP-WC system can be proposed as a valid tool for operational use (see further details therein Settimi's et al ref. [2015]).

Figure 4.c reproduces, with reference to the caption of Figure 4.a, the corresponding semi-logarithmic plots of the non-deviative absorption L [dB], according to the COMPLEIK model, as a function of the frequency f [MHz].

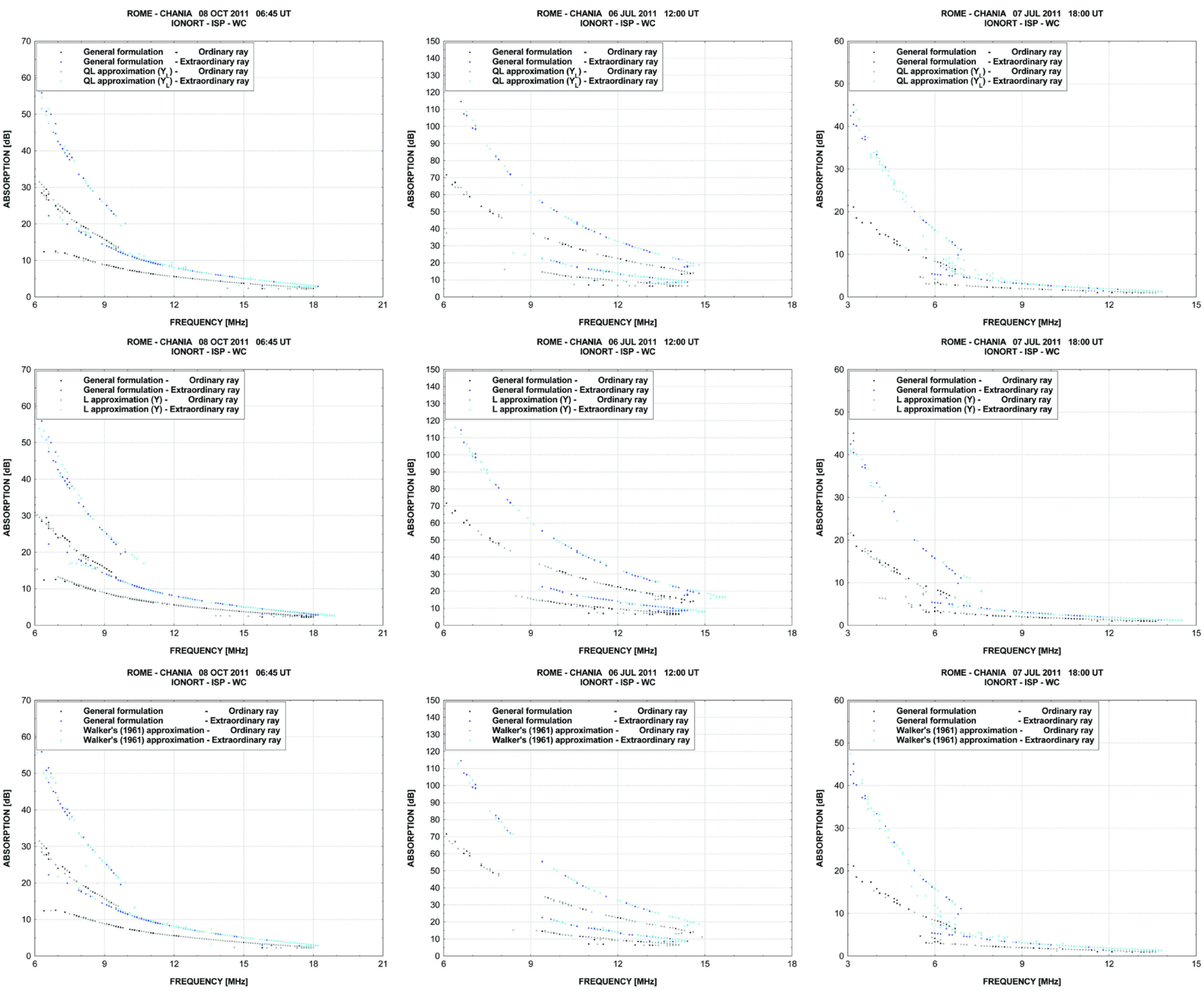


Figure 4.c. Corresponding semi-logarithmic plots of the non-deviative absorption L [dB], according to the COMPLEIK model, as a function of the frequency f [MHz].

Both the ordinary and extraordinary rays are modelled by: Appleton-Hartree's general formulation; moreover, the QL approximation (by Y_L) and the L approximation (by Y); and, finally, Walker's [1961] approximation respectively.

In Figure 4.c, as a note, each absorption profile is composed by: 1) a pair of ordinary and extraordinary traces corresponding to the ionospheric F1-F2 layers at high altitudes ($h > 150$km) and characterized by a low absorption coefficient ($L \leq 20$ dB); 2) another pair of ordinary and extraordinary traces corresponding to the E-layer at bottom altitude, 90 km $< h \leq 150$ km, and with a higher absorption coefficient ($L >> 20$ dB) [MacNamara, 1991].

As comments of both Figure 4.b, reproducing the synthesized oblique ionograms, and Figure 4.c, reproducing the semi-logarithmic plots of the integral absorption L [dB] vs the frequency f [MHz], the L approximation (Y), when modelling both the ordinary and extraordinary rays, could be evaluated reliable just for a radio propagation at enough low frequencies, i.e. $f \ll$ MUF.

Note, analogously to Figure 3.a, that, under the L approximation (Y), the critical penetration frequency is linearly dependent just on the geomagnetic field amplitude parameter Y, reason why the deviance of radio propagation could be uniformly distributed throughout the whole frequency band.

Moreover, conforming to Figure 1, independently from Booker's [1935] rule is considering the sign change in the range $X \ll 1$, a good eye may catch sight that both Walker's and QL (Y_L) approximations almost superimpose to the general formulation, when modelling just the ordinary ray, throughout the whole band of frequency, i.e. $f \leq$ MUF.

Finally, conforming to Figure 1, independently from Booker's [1935] rule is considering the sign change in the range $X \ll 1$, a good eye may catch sight that Walker's approximation is slightly more accurate than the QL approximation (Y_L) in superimposing to the general formulation, when modelling just the extraordinary ray, especially around the maximum usable frequency, i.e. $f \approx$ MUF.

Note, analogously to Figure 3.b, as regards a general ordinary or extraordinary ray, the L approximation (Y) is not at all reliable, instead Walker's approximation is anyhow more accurate than the QL approximation (Y_L). Conforming to Figure 1, independently from Booker's [1935] rule is considering the sign change in the range $X \ll 1$, Walker's approximation almost superimposes the QL approximation (Y_L), when modelling the ordinary ray; while, Walker's approximation is slightly more accurate than the QL approximation (Y_L), when modelling the extraordinary ray.

4.3 Vertical sounding and absorption

Let us consider a "short" radio link, with a single ionospheric reflection (1 hop), between the transmitter (TX), located in Rome (RM), Italy (lat_{TX} = 41°53′35″N, lon_{TX} = 12°28′58″E) and the receiver (RX), located in Montelibretti (RM), Italy (lat_{RX} =42°08′00″N, lon_{RX} = 12°44′00″E). International Reference Ionosphere (IRI)-2007 model allows computing the E plasma critical frequency f_oE, on any date and time, relative to the RX station. IRI (or ISP) model generates a 3-D (even real-time) electron density grid of the ionosphere over Mediterranean area, in order to synthesize "quasi" vertical ionograms of the "short" radio link between TX and RX stations, at a "short" distance away.

Figure 5.a reproduces the vertical ionogram recorded by AIS-INGV digital ionosonde [Zuccheretti et al., 2003] over Montelibretti (RM) station on 23 February 2010 at 07:30 UT. In the vertical ionogram, the frequency, which is

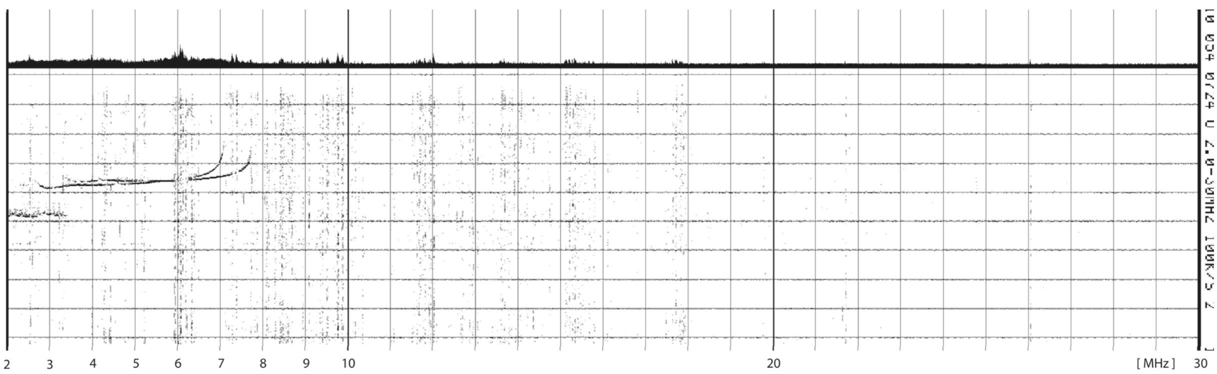


Figure 5.a. Vertical ionogram recorded by AIS-INGV digital ionosonde over Montelibretti (RM) station on 23 February 2010 at 07:30 UT.

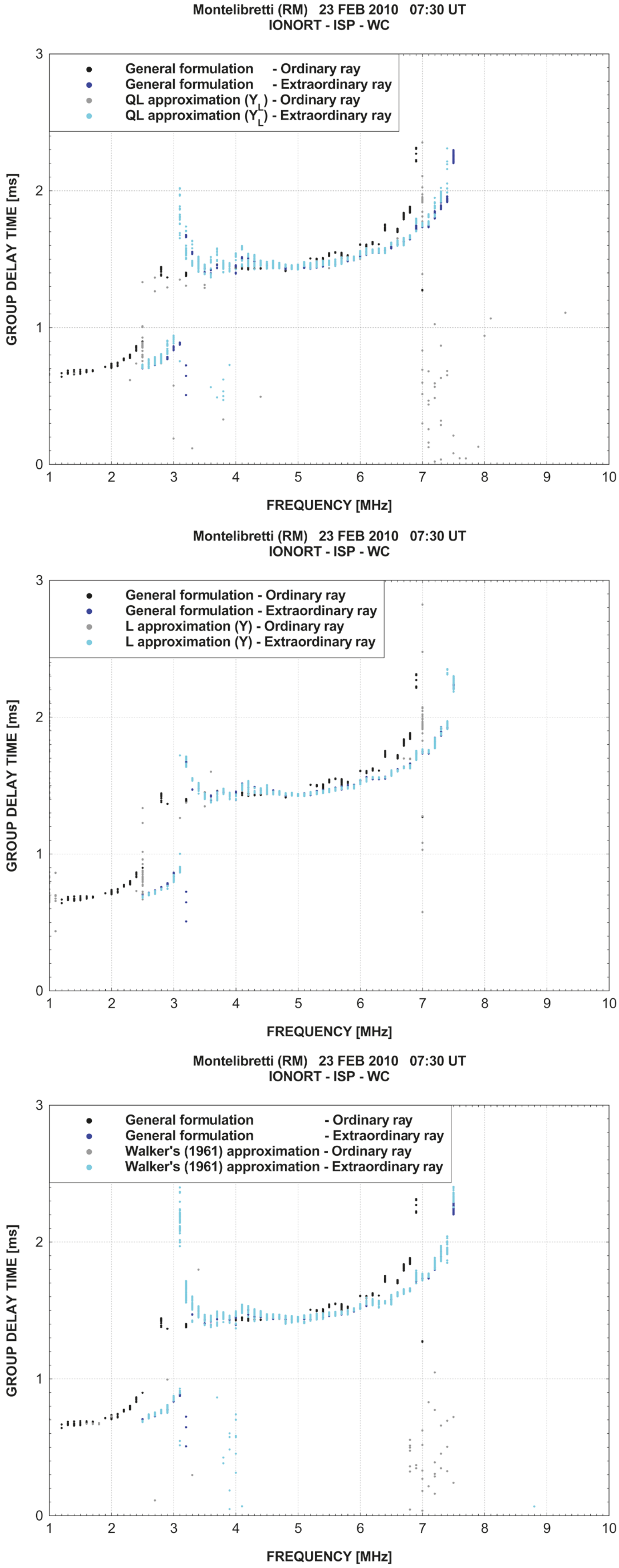


Figure 5.b. Corresponding "quasi" vertical ionograms along Rome (TX) – Montelibretti (RX) "short" radio link, synthesized by IONORT-ISP-WC system.

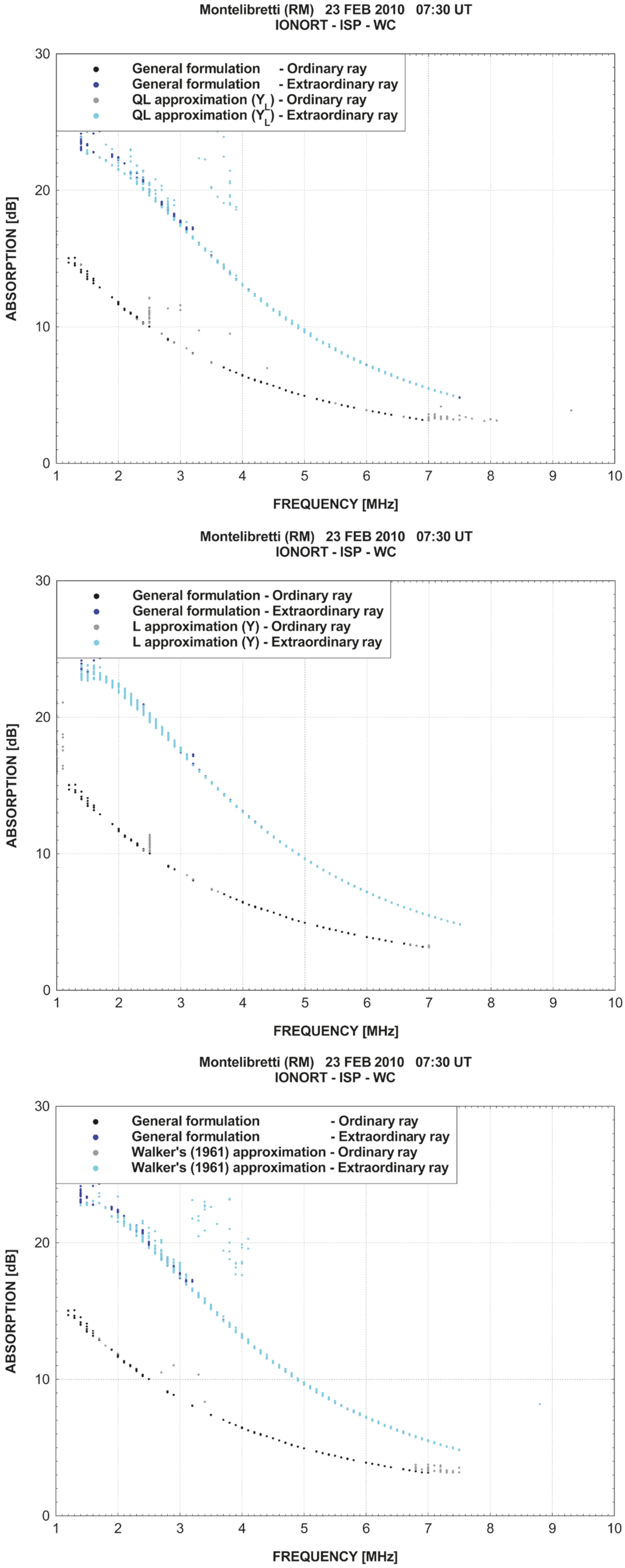


Figure 5.c. Corresponding semi-logarithmic plots of the non-deviative absorption L [dB], according to the COMPLEIK model, as a function of the frequency f [MHz].

the sole physical magnitude to be relied upon, runs from 2 MHz, then each notch is equal to 1 MHz, and finally the thick cuts are 10, 20, 30 MHz.

Figure 5.b reproduces, with reference to the caption of Figure 5.a, the corresponding "quasi" vertical ionograms along Rome (TX) – Montelibretti (RX) "short" radio link, synthesized by IONORT-ISP-WC system [Settimi et al., 2013a; 2015]. In the "quasi" vertical ionograms, both the ordinary and extraordinary traces, computed by taking IGRF-12 geomagnetic induction field and Jones-Stephenson's [1975] double exponential profile of electron collision frequency into account, are shown. For one ionospheric reflection (1 hop path), a nested loop cycle was iterated with angular frequency set from 1 MHz to 30 MHz of step 0.1 MHz, moreover the azimuth angle was set from 37° to 39° of step 0.1°, the elevation angle set from 0° to 90° of step 0.1°, and finally the RX range accuracy to 0.1 %.

As in Figure 4.b even in Figure 5.b, as a note, the jitter noise is due to some numerical instabilities, which could be offset by adjusting the integration step of the ray path length as a function of the local value for the phase refractive index. However, the feedback would not be so satisfactory for the "quasi" vertical ionograms synthesized by the discrete IONORT-ISP-WC system, as would be expected using an analytical Chapman's [1931a; 1931b] electron density.

Figure 5.c reproduces, with reference to the caption of Figure 5.a, the semi-logarithmic plots of the non-deviative absorption L [dB] corresponding to the ionograms of Figure 5.a, previewed by the COMPLEIK model, as a function of the frequency f [MHz] [Settimi et al., 2013b; 2014a; 2014b]. As in Figure 4.a, both the ordinary and extraordinary rays are modelled by: Appleton-Hartree's general formulation; moreover, the QL approximation (by Y_L) and the L approximation (by Y); and, finally, Walker's [1961] approximation.

In Figure 5.c, as a note, each absorption profile is composed by a pair of ordinary and extraordinary traces corresponding to the ionospheric F1-F2 layers at high altitudes ($h > 150$ km) and characterized by a low absorption coefficient ($L \leq 20$ dB) [McNamara, 1991].

Since TX station, i.e. Rome ($lat_{TX} = 41.893056°$ N, $lon_{TX} = 12.482778°$ E), and RX station, i.e. Montelibretti ($lat_{RX} = 42.133333°$ N, $lon_{RX} = 12.733333°$ E), are distant 33.8077 km, it can be deduced that a RX range accuracy of 0.1 %, corresponding to latitude and longitude errors respectively of 0.042133333° and 0.012733333°, involves the distance errors in latitude 1.42443 km and longitude 0.430485 km, which can be considered as a tolerable accuracy for the "quasi" vertical ionograms of "short" radio links over the TX and RX stations. Moreover, even reducing the step of elevation angle down to less than 0.1°, anyway the advantage of transforming the *discrete* points into a *continuum* line is not reached for the traces of both ordinary and extraordinary rays, and, as an additional disadvantage, the algorithm becomes time-consuming and low efficient. Finally, each profile of non-deviative absorption consists in a string of vertical segments, especially at the lower frequencies, and much more for the extraordinary ray rather than the ordinary ray: indeed, once fixed the low radio frequency, then it may occur that some ionospheric horizontal gradients allow a short series of close-range azimuth angles linking the TX and RX stations.

As depicted in Figure 5.b, even Figure 5.c confirms that the L approximation (Y) looks reliable in overall propagation conditions: indeed, regarding the ordinary ray, it seems much more accurate than both QL (Y_L) and Walker's approximations, especially in proximity of actual ordinary reflection; instead, regarding the extraordinary ray, it seems slightly more accurate than both QL (Y_L) and Walker's approximations, especially next to the lowest frequencies.

Comparing Figures. 5.b and 5.c highlights that the L approximation (Y) should be carefully applied for the ionospheric vertical sounding simulations: indeed, regarding the ordinary ray, it seems more accurate in order to synthesize the profiles of absorption than the corresponding "quasi" vertical ionograms, especially in proximity of actual ordinary reflection; instead, regarding the extraordinary ray, it seems equally accurate for both the absorption profiles and the vertical ionograms, throughout the ray-tracing. Concluding, the L approximation (Y) is generally more accurate in order to synthesize the absorptions than the ionograms.

5. Results and analysis

When the electromagnetic wave propagates in the ionospheric plasma, starts from areas where the electron density is zero ($X = 0$) and penetrates in areas with increasing electron density, to meet layers where the electron density is such as to take place reflection ($\mu = 0$).

It is therefore interesting, in Figure 1, to compare the curves, plotting the μ_{ord} and μ_{ext} obtained by the Appleton-Hartree's general formulation with the μ_{ord} and μ_{ext} obtained by the different QL(Y_L), L(Y) and Walker's [1961] approximations, in the range of X that goes from $X = 0$ to the reflection ($\mu = 0$).

From this comparison, it should then be noted that Walker's approximation works generally better than both the QL(Y_L) and L(Y) approximations.

Concerning the ordinary ray, as long as the Booker's [1935] rule imposes the change of sign, the QL(Y_L), L(Y) and Walker's approximations model quite well the propagation till the reflection. Conversely, when the Booker's rule does not require the change of sign, the different approximate formulas fail close to the reflection level, do not providing the correct frequencies of reflection, i.e. $X = 1$.

Concerning the extraordinary ray, regardless the fact that the Booker's rule imposes the change of sign, the QL(Y_L) and L(Y) [Walker's] approximation cannot [can] be used to compute the ray path in non deviative conditions till very close to reflection level, working [not so] badly in the range from $X = 0$ to $X \leq 1 - Y$.

This can be inferred from the graphs of Figure 1, and it is confirmed from ionospheric high frequencies (HF) ray-tracing (see Figs. 3.a-b) and oblique ionogram (see Figs. 4.a-c) simulations.

It is worth to note that these statements are generally true because IONORT program, simulating the radio propagation in the ionosphere, runs in operative conditions such that Booker's rule does not impose a change of sign, occurring when Booker's critical frequency ω_c and the collision frequency ν satisfy the inequality $\omega_c/\nu > 1$.

However, these results correspond to noteworthy achievements, which suggest the use of QL approximation in computing the absorption, for the ordinary ray only in non deviative conditions, and for the extraordinary ray in both deviative and non deviative conditions. In order to compute the ray path, only if Booker's rule does not impose the change of sign, then the QL approximation fails close to reflection level for the ordinary ray, assuming the propagation as occurring in a straight line, and can be generally used till reflection level for the extraordinary ray.

Such as analysis can be schematically summarized by a logical flowchart illustrating how IONORT performs simulations satisfying operatively the "strong", or "weak" QL conditions, either Walker's approximation (Figure 6).

Even vertical ionogram simulations (see Fig. 5.a-c) confirm that: regarding the ordinary ray, all the QL (by Y_L) and L (by Y) and Walker's approximations are not at all reliable, especially in proximity of actual ordinary reflection, i.e. $X \approx 1$; instead, regarding the extraordinary ray, the QL approximation (by Y_L) is not so accurate in proximity of extraordinary reflection, i.e. $X \approx 1 - Y$, and the L (by Y) approximation is slightly more accurate the Walker's approximation, especially in proximity of actual extraordinary reflection, i.e. $X \approx 1 - Y$.

Concluding: the ordinary traces computed according to the different QL (by Y_L) and L (by Y) and Walker's approximations are not in line with the trace obtained by the Appleton-Hartree's general formulation; conversely, the extraordinary traces computed according to Eq. (4) [Eqs. (3) and (6) do not] correspond generally to the results given by Eq. (1).

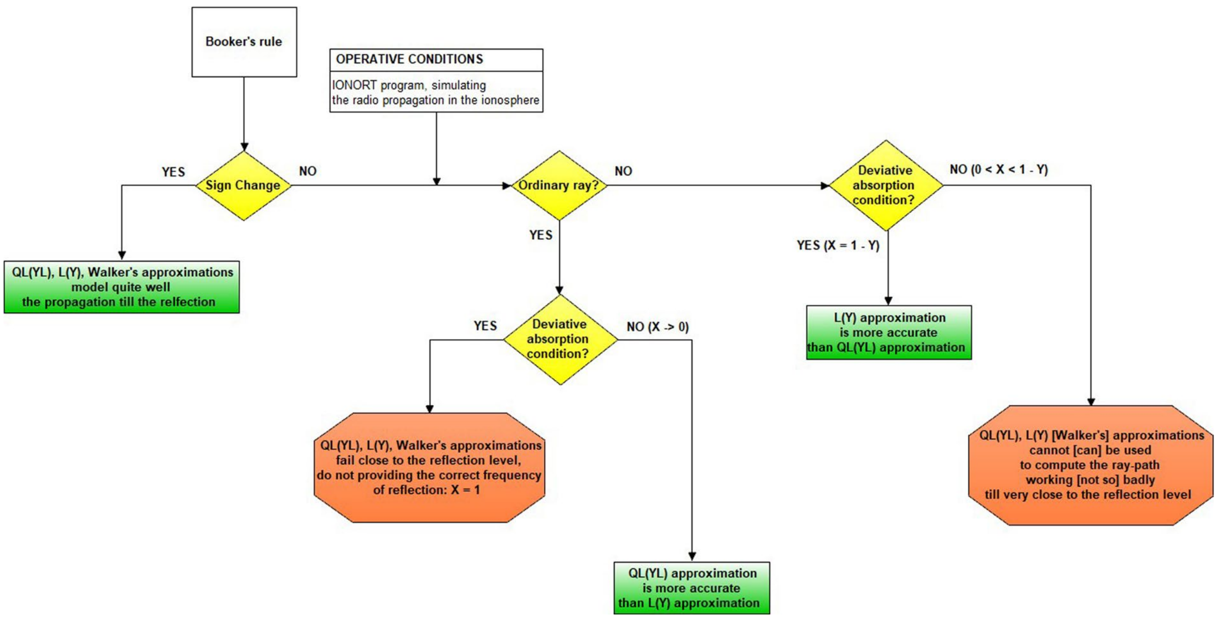


Figure 6. A logical flowchart illustrating how IONORT performs simulations satisfying operatively the "strong", or "weak" (Quasi-Longitudinal) QL conditions, either Walker's approximation.

6. Discussion and conclusions

For the phase refraction index of high frequency (HF) waves in the ionospheric medium exists a well-established theory. However, under the Quasi-Longitudinal (QL) conditions, scientific literature presents various formulas that are not equivalent and that, in some cases, give rise to wrong results. In the present study, further consequences of Booker's [1935] rule were discussed, illustrating the validity ranges of the above-mentioned approximate formulas; and the different regimes for applying such QL formulas were described, along with the consequences in simulating the ionospheric HF ray-tracing, oblique and vertical sounding, and absorption.

As a rule, Appleton-Hartree's general formulation states formally that the ionospheric medium flexes the HF radio wave, when satisfying the law of reflection at the critical frequency of penetration [Budden, 1988]. Indeed, the physical phenomenon could be analysed in terms of an in-continuum refraction law throughout the whole frequency band, before reaching the critical penetration condition. Under the QL approximation, the ordinary [extraordinary] ray, before being reflected at higher [lower] altitudes under the actual penetration condition $X \approx 1$ [$X = 1 - Y$], propagates almost in vertical [parallel] to the direction of geomagnetic induction field, i.e. θ [Budden, 1988]. Therefore, the QL approximation should provide a model for the ordinary ray less accurate than the extraordinary ray, which fulfils more properly (before its actual reflection) the condition for the geomagnetic field angle: $\theta \approx 0$.

Forcing just the Longitudinal (L) approximation, the input data on geomagnetic field angle are essentially dropped ($\theta \approx 0$), and even the real part of Eq. (4) denominator is erroneously maximized ($1 \pm Y_L = 1 \pm Y \cos\theta \leq 1 \pm Y$), so that a wider gap is interposed between the frequency ω and the gyro-frequency ω_B: L (by Y) approximation should be [not] reliable for all the "quasi" vertical [oblique] soundings along a "short" [long] radio link between the TX and RX stations at a "short" [some] distance away, since both the ordinary and extraordinary rays [do not] fulfil (throughout their ray tracing) the condition on geomagnetic field angle: $\theta \approx 0$.

Walker's [1961] and QL (Y_L) approximations should be reliable, much more modelling the extraordinary ray rather than the ordinary ray, for all the oblique and vertical soundings along a radio link between two stations, since the extraordinary ray fulfils more properly than the ordinary ray (throughout its ray tracing) the condition on geomagnetic field angle: $\theta \approx 0$.

Appendix has underlined that: the L approximation (appearing Y) [whence Eq. (A.4)], superimposes to Walker's approximation on almost the whole range of plasma frequency parameter X less than around the step point $X \approx 1$, as long as Booker's rule imposes the sign change, consistently with a low geomagnetic field angle θ. Moreover, the L approximation (by Y) is accurate just on a short range of parameter X such that $X \ll 1$, if Booker's rule does not impose a sign change, due to a middle value of θ. Finally, the L approximation (Y) is not at all reliable on any range of X, if Booker's rule does not consider a sign change, when raising to a high θ.

7. Appendix

The L approximation (appearing Y) (4) can be merged with Walker's [1961] approximation (6), when their denominators become equivalent:

$$\pm Y \simeq -\frac{Y_T^2}{2\left(1-X-iZ\right)} \pm Y_L = -\frac{Y_T^2}{2}\frac{1-X}{\left(1-X\right)^2+Z^2} \pm Y_L - i\frac{Y_T^2}{2}\frac{Z}{\left(1-X\right)^2+Z^2}, \tag{A.1}$$

and, consequently, the "strong" and "weak" QL conditions (2) and (5) can be cast into a so-called "Y–Walker's" QL condition, if, with reference to Eq. (A.1), the real field Y of the first member (on left side) is approximately approaching the complex field of the second member (in right side), even in the worst case when selecting the plus sign (+):

$$\begin{aligned} 0 \leq \frac{Y_T^2}{2}\frac{Z}{\left(1-X\right)^2+Z^2} \ll -\frac{Y_T^2}{2}\frac{1-X}{\left(1-X\right)^2+Z^2} + Y_L \Leftrightarrow \frac{Y_T^2}{2}\frac{\left(1-X\right)+Z}{\left(1-X\right)^2+Z^2} \ll Y_L \\ \Leftrightarrow \frac{Y_T^2}{2Y_L} \ll \frac{\left(1-X\right)^2+Z^2}{\left(1-X\right)+Z} \end{aligned}. \tag{A.2}$$

Once the definition of geomagnetic field amplitude and angle parameters is inserted into Eq. (A.1):

$$\begin{cases} Y_{\mathrm{L}} = Y\cos\theta \\ Y_{\mathrm{T}} = Y\sin\theta,\ Y \neq 0 \end{cases}, \tag{A.3}$$

under the QL condition "Y–Walker's" (A.2), Eq. (A.1) can be reduced to an implicit relation of dispersion for the ionospheric magneto-plasma [Taylor, 1933; 1934], which links its parameters Z, X, Y and θ, i.e.:

$$Y \simeq \left|1 - X + \frac{Z^2}{1-X}\right| \Big/ \cos^2\frac{\theta}{2}. \tag{A.4}$$

The L approximation (appearing Y) [whence Eq. (A.4)], superimposes to Walker's approximation on almost the whole range of plasma frequency parameter X less than around the step point $X \approx 1$, as long as Booker's rule [1935] imposes the sign change, consistently with a low geomagnetic field angle θ. Moreover, the L approximation (by Y) is accurate just on a short range of parameter X such that $X \ll 1$, if Booker's rule does not impose a sign change, due to a middle value of θ. Finally, the L approximation (Y) is not at all reliable on any range of X, if Booker's rule does not impose a sign change, when raising to a high θ (see Fig. 1).

If the electron collision effects can be neglected in the magneto-plasma, under the condition for the collision frequency parameter:

$$Z \ll |1-X|, \tag{A.5}$$

then the dispersion relation (A.4) is simplified, linking only the parameters X, Y and θ, i.e.:

$$Y \approx |1-X| \Big/ \cos^2\frac{\theta}{2}. \tag{A.6}$$

In case of a QL non-deviative propagation, under the conditions for the plasma frequency parameter:

$$X \leq 1, X \to 0, \tag{A.7}$$

the implicit dispersion relation (A.6) can be further simplified, so that Y is explicitly expressed as an increasing function of θ in a closed form, i.e.:

$$Y \approx (1-X) \Big/ \cos^2\frac{\theta}{2} \sim 1 \Big/ \cos^2\frac{\theta}{2}. \tag{A.8}$$

In case of a whistler's branch [Davies, 1990], under the related conditions:

$$X > 1, Y \gg 1, X/Y \gg 1, \tag{A.9}$$

to be imposed on the phase refractive index:

$$n^2 \approx 1 - \frac{X}{1 \pm Y} \sim 1 + \frac{X}{Y} \sim \frac{X}{Y}, \tag{A.10}$$

and, once the definition of X and Y parameters is inserted:

$$\begin{cases} X = \left(\dfrac{\omega_{\mathrm{N}}}{\omega}\right)^2 \\ Y = \dfrac{\omega_{\mathrm{H}}}{\omega} \end{cases}, \tag{A.11}$$

into Eq. (A.6), reported expressly with the minus sign (−), as holding Eq. (A.9), i.e.:

$$Y \approx -\left(1 - X\right) \Big/ \cos^2 \frac{\theta}{2}, \tag{A.12}$$

then it results the phase refractive index, when varying weakly in space, condition for which the refractive index can be matched to the first derivative of X relative to Y, as a decreasing function of θ, i.e. [Bianchi, 1990]:

$$n^2 \sim \frac{X}{Y} \cong \frac{dX}{dY} = 2\frac{\omega_N^2}{\omega\omega_H} \approx \cos^2 \frac{\theta}{2}. \tag{A.13}$$

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