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APPENDIX TO

ELECTROKINETIC EFFECT PROVIDED BY LONG OCEANIC WAVES COMING ON SHORE

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The Laplace transform of equation (10) with respect to the variable x_1 is given by

$$\tilde{p}'' + 2q\cos\alpha\,\tilde{p}' + \left(q^2 + \frac{i\omega\sin^2\alpha}{D}\right)\tilde{p} = qp\left(0, z_1\right) + \frac{\partial p\left(0, z_1\right)}{\partial x_1} + 2\cos\alpha\frac{\partial p\left(0, z_1\right)}{\partial z_1}. \qquad \text{(A1)}$$

Here q is the Laplace transform parameter, the primes denote the derivatives with respect to z_1 and $\tilde{p}\left(q, z_1\right)$ stands for Laplace transform of the function $p_p\left(x_1, z_1\right)$:

$$\tilde{p}\left(q, z_1\right) = \int_0^\infty p_p\left(x_1, z_1\right)\exp\left(-qx_1\right)dx_1, \quad \left(\operatorname{Re} q > 0\right). \qquad \text{(A2)}$$

Substituting equations (11) and (13) for the functions $p_p\left(0, z_1\right)$ and $\partial p_p\left(0, z_1\right)/\partial x_1$ into equation (A1), yields

$$\tilde{p}'' + 2q\cos\alpha\,\tilde{p}' + \left(q^2 + \frac{i\omega\sin^2\alpha}{D}\right)\tilde{p} = f\left(q, z_1\right). \qquad \text{(A3)}$$

Here we made use of the following designation:

$$f = \rho g\eta_m\left\{q\,\mathrm{J}_0\left(2\left\{sz_1\cos\alpha\right\}^{1/2}\right) + \left(\frac{s\cos\alpha}{z_1}\right)^{1/2}\left(\zeta - \cos\alpha\right)\mathrm{J}_1\left(2\left\{sz_1\cos\alpha\right\}^{1/2}\right)\right\}. \qquad \text{(A4)}$$

First, we consider the homogeneous differential equation (A3), assuming formally that the right side of this equation is zero. The roots of the corresponding characteristic equation are given by

$$\lambda_{1,2} = -q\cos\alpha \pm i\sin\alpha\left(q^2 + i\omega/D\right)^{1/2}. \qquad \text{(A5)}$$

When applying the inverse Laplace transform, the integration is performed in a complex plane $q = q_1 + iq_2$ along a vertical axis parallel to the imaginary axis iq_2. Let us cut the complex plane q through the branch points $q = \pm(i\omega/D)^{1/2}$ of the functions λ_1 and λ_2 in such a way that the inequalities $\operatorname{Re}\lambda_1 < 0$ and $\operatorname{Re}\lambda_2 > 0$ hold true on the integration path. In the extreme case, when $|q_2| \to \infty$ one can simplify equation (A5) for λ_1 and λ_2:

$$\lambda_{1,2} \approx -iq_2 \cos\alpha \mp \sin\alpha |q_2|. \tag{A6}$$

Here the upper sign minus corresponds to the function λ_1, and the lower sign plus corresponds to the function λ_2.

Using the method of variation of constants, we find a general solution of the inhomogeneous differential equation (A3)

$$\tilde{p}(q,z_1) = F_1(q,z_1) - F_2(q,z_1) + C_1 \exp(\lambda_1 z_1) + C_2 \exp(\lambda_2 z_1), \tag{A7}$$

where

$$F_{1,2}(q,z_1) = \frac{1}{\lambda_1 - \lambda_2} \int_0^{z_1} f(q,z') \exp\{\lambda_{1,2}(z_1 - z')\} dz'. \tag{A8}$$

The undetermined coefficients C_1 and C_2 can be found from the boundary condition $\tilde{p}(q,z_1) = 0$ and the boundedness condition for the function $\tilde{p}(q,z_1)$ in the extreme case of $z_1 \to \infty$. As a result, we obtain:

$$\tilde{p}(q,z_1) = F_1(q,z_1) - F_2(q,z_1) + A(q)\{\exp(\lambda_2 z_1) - \exp(\lambda_1 z_1)\}, \tag{A9}$$

where

$$A(q) = \frac{1}{\lambda_1 - \lambda_2} \int_0^{\infty} f(q,z') \exp(-\lambda_2 z') dz'. \tag{A10}$$

In order to derive an approximate solution of the problem for the case of small depths z, we first expand equation (A9) for the function $\tilde{p}(q,z_1)$ in a power series of z_1, preserving only the first non-vanishing term:

$$\tilde{p}(q,z_1) \approx z_1(\lambda_2 - \lambda_1) A(q) = -z_1 \int_0^{\infty} f(q,z') \exp(-\lambda_2 z') dz'. \tag{A11}$$

Next, we substitute equation (A4) for the function f into equation (A11). After replacing the integration variable: $z' = x'^2$, the integral in equation (A11) is reduced to tabular integrals [Grandshteyn and Ryzhik, 2007, p. 698]. As a result, we obtain:

$$\tilde{p}(q,z_1) \approx -z_1 \rho g \eta_m \left\{ \frac{q}{\lambda_2} \exp\left(-\frac{s\cos\alpha}{\lambda_2} \right) + (\zeta - \cos\alpha) \left[1 - \exp\left(-\frac{s\cos\alpha}{\lambda_2} \right) \right] \right\}. \qquad \text{(A12)}$$

The pressure of the pore fluid can be found by applying the inverse Laplace transform to equation (A12):

$$p_p(x_1,z_1) = \frac{1}{2\pi i} \int_{q_0 - i\infty}^{q_0 + i\infty} \tilde{p}(q,z_1) \exp(q x_1)\, dq, \quad (q_0 > 0), \qquad \text{(A13)}$$

Substituting equation (A12) for $\tilde{p}(q,z_1)$ into equation (A13) and taking into account that $x_1 \approx x$ near the boundary between the porous medium and the atmosphere, we get

$$p_p(x,z) \approx -\frac{\rho g z \eta_m}{2\pi i \sin\alpha} \left\{ Q_1 + (\cos\alpha - \zeta)\left[Q_2 - 2\pi\delta(x) \right] \right\}. \qquad \text{(A14)}$$

In the next we will ignore the Dirac delta function $\delta(x)$, since the approximate solution (A14) is valid in the region $x \gg z$, where this function is zero. Here we made use of the following abbreviations:

$$Q_1 = \int_{q_0 - i\infty}^{q_0 + i\infty} \frac{q}{\lambda_2(q)} \exp\left(-\frac{s\cos\alpha}{\lambda_2(q)} + qx \right) dq, \quad Q_2 = \int_{q_0 - i\infty}^{q_0 + i\infty} \exp\left(-\frac{s\cos\alpha}{\lambda_2(q)} + qx \right) dq. \qquad \text{(A15)}$$

To perform integration in equation (A15), we introduce a new integration variable $q_2 = -iq$. For low frequencies, when $(\omega/D)^{1/2} \ll s, x^{-1}$, one can use an approximate equation (A6) for λ_2. Substitution q_2 into equation (A6) gives $\lambda_2 \approx -iq_2 \exp(\pm i\alpha)$ where the plus sign in the exponent corresponds to positive values of q_2 while the minus corresponds to negative q_2. Substituting λ_2 into equation (A15) and rearranging, we come to

$$Q_1 \approx -2i \operatorname{Re} \exp(-i\alpha) N, \quad Q_2 \approx 2i \operatorname{Re} N, \qquad \text{(A16)}$$

where

$$N = \lim_{\varepsilon \to 0} \int_0^\infty \exp\left\{ -\frac{is\cos\alpha}{q_2} \exp(-i\alpha) + iq_2 (x + i\varepsilon) \right\} dq_2. \qquad \text{(A17)}$$

In order for the integral (A17) to be convergent a small value ε is formally added to the second term in curly brackets. In this case the real part of the function in curly brackets is negative. Then the integral N is reduced to a tabular integral [Grandshteyn and Ryzhik, 2007], which can be rearranged to the form:

$$N = 2i\exp\left(\frac{i\alpha}{2}\right)\left(\frac{s\cos\alpha}{x}\right)^{1/2}\mathrm{K}_1\left(2\exp\left\{\frac{i\alpha}{2}\right\}\left\{xs\cos\alpha\right\}^{1/2}\right), \tag{A18}$$

where K_1 is the modified first-order Bessel function. Substituting equations (A16) and (A18) into the equation (A14), we obtain an approximate expression for the fluid pressure variation in a porous medium.