Convection of a micropolar fluid with stretch

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Summary. — As a model for the Bénard convection in the asthenosphere the problem of the hydrodynamic stability of an infinite horizontal layer is calculated. The layer consists of a micropolar fluid *with stretch*. The field equations for the velocity vector, microrotation vector, microstretch, microinertia, density, temperature, and pressure form a system of eleven partial differential equations for the determination of eleven unknown scalar functions. We succeed in decoupling the system and reducing the problem to an ordinary differential equation. The analytical solution can be given for the special case of a micropolar Boussinesq fluid.

Zusammenfassung. — Als Modell für die Bénard-Konvektion in der Asthenosphäre wird das hydrodynamische Stabilitätsproblem einer unendlichen horizontalen Schicht berechnet. Die Schicht besteht aus einer kompressiblen mikropolaren Flüssigkeit. Die Feldgleichungen für den Geschwindigkeitsvektor, den Mikrorotationsgeschwindigkeitsvektor, die Mikrodeformation, das Mikroträgheitsmoment, die Dichte, die Temperatur und den Druck bilden ein System von elf partiellen Differentialgleichungen zur Bestimmung von elf unbekannten skalaren Funktionen. Es gelingt, das System zu entkoppeln und das Problem auf eine gewöhnliche Differentialgleichung zu reduzieren. Für den Spezialfall einer mikropolaren Boussinesq-Flüssigkeit kann die analytische Lösung angegeben werden.

Riassunto. — Il problema della stabilità idrodinamica di uno strato infinito orizzontale viene trattato come un modello per la convenzione Bénard nell'astenosfera. Lo strato è formato da un fluido micropolare "with stretch". L'insieme delle equazioni per il vettore velocità, il vettore microrotazione, la microdeformazione, la microinerzia, la densità, la temperatura e la pressione, forma un sistema di 11 equazioni differenziali parziali che

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serve a determinare le 11 funzioni scalari incognite. Si raggiunge lo scopo decuplicando il sistema e riducendo il problema ad un'equazione differenziale ordinaria. Si può dare la soluzione analitica solo nel caso di un fluido micropolare Boussinesq.

Introduction

When investigating flow phenomena in the Earth's mantle, a Newtonian fluid is mostly used as constitutive equation. This assumption is fully justified as an approximation and has often been used successfully [3, 5, 9, 11, 14]. From shear tests of possible mantle

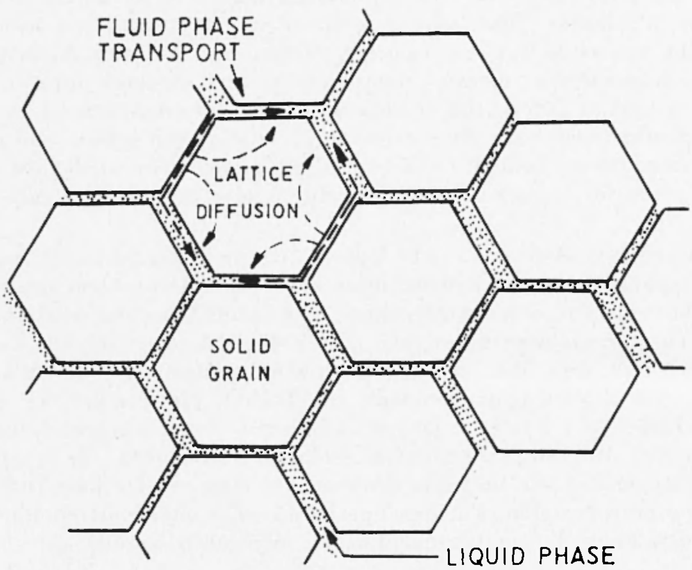

Fig. 1 – Schematic representation of the geometry of the two phases of low-velocity layer material. After Stocker and Ashby [12].

rock [8], from the postglacial uplift of Fennoscandia [10], and from theoretical considerations [18] it is concluded that a power law fluid with the exponent 3 is a more realistic model for the *solid* portions of the mantle. For the corresponding convection based on solid creep a theory has been developed (17). In another approach (15), used to gain information on the convection current pattern in the Earth's mantle, the question as to the constitutive equation is evaded. Some simple assumptions are made regarding the kinematics of the stream lines which appear to be plausible from the point of view of fluid mechanics. These assumptions, together with the geometry of the mantle, lead to certain possible modes of flow which would create a topography on the surface of the Earth which is similar to the ob-

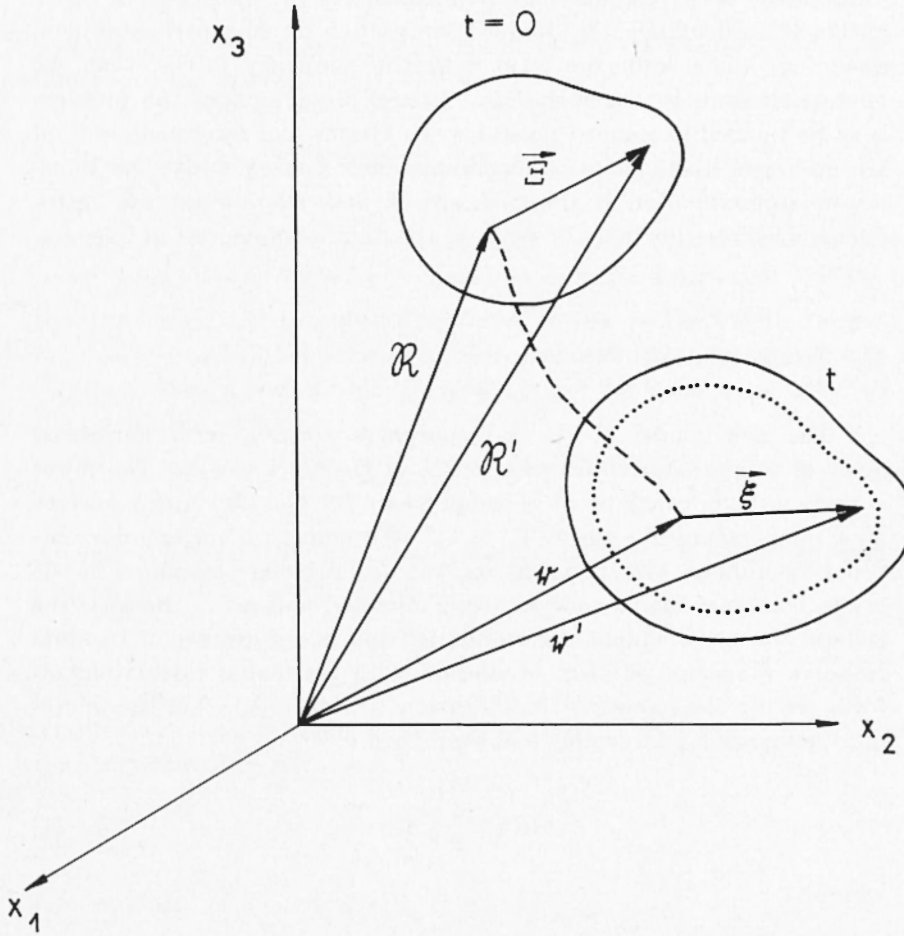

Fig. 2 – Translation, rotation and isotropic microstretch of an element of a micropolar fluid with stretch.

served one. Recently, the low-velocity layer of the upper mantle has been assumed to be partly molten ([1]). If the geometrical connection between the two phases exists in the form (see Fig. 1) suggested by Stocker and Ashby ([12]), a micropolar fluid may be assumed as a constitutive equation. In such a medium, in addition to the three translatory degrees of freedom of conventional continuum mechanics, three rotational degrees of freedom are assigned to each spatial point with the help of which the rotation of the solid grains of Fig. 1 may be described. While some authors have already tried to work with micropolar *elastic* media (e. g., Teisseyre ([13]), Boschi ([2])), Cosserat *fluids* have been introduced only recently ([16]) into geophysics. The author ([16]) calculated the Bénard convection in the asthenosphere, assuming, apart from the term with the buoyancy forces, that the micropolar fluid is *incompressible*. In the present paper the problem is to be tackled in a more general way. Grains and intermediate fluid are no longer assumed to be incompressible. Consequently, the Boussinesq approximation is dropped, *stretch* and *microstretch* are introduced, which results in an increase of the number of degrees of freedom.

1. – Governing equations

The new model of the asthenosphere consists of a horizontal layer of a micropolar fluid *with stretch* of the thickness h. The lower surface is kept constant at a temperature T_0 and the upper surface at a temperature T_1, where $T_0 > T_1$. We employ a rectangular cartesian coordinate system x_1, x_2, x_3, the origin being positioned in the lower boundary plane and x_3 being directed upwards. Because the general theory of simple microfluids has too many degrees of freedom to solve a special problem of motion with justifiable calculation efforts, we use the following simplifications (Eringen ([7])). Let the microinertia tensor i_{kl} have the following form:

$$i_{kl} = \frac{1}{2}\, j\, \delta_{kl} \qquad [1]$$

where

$$\delta_{kl} = \begin{cases} 1 & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases} \qquad [2]$$

and j is a scalar quantity, i. e., the fluid is microisotropic. Let the gyration tensor n_{kl} have 4 independent scalar functions instead of 9:

$$n_{kl} = n\,\delta_{kl} + e_{klr}\,n_r \qquad [3]$$

where

$$e_{klr} = \begin{cases} 1 & \text{for } (klr)^{cycl.} = (1\,2\,3) \\ -1 & \text{for } (klr)^{cycl.} = (1\,3\,2) \\ 0 & \text{for other cases} \end{cases} \qquad [4]$$

n or n_r = microrotation vector and n = microstretch. The significance of these quantities is demonstrated in Fig. 2. In the time t, point **R** of the fluid moves to **r**, point **R'** to **r'**. The vector $\vec{\xi}(t)$ or $\xi_k(t)$ represents the micromotion:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}.\nabla\right)\vec{\xi} = n_{kl}\,\xi_k \mathbf{e}_l = n\,\xi_l\,\mathbf{e}_l + \mathbf{n} \times \vec{\xi}. \qquad [5]$$

Vectors are indicated by bold-face letters or by arrows over the letters. **v** or v_r = velocity, $\mathbf{e}_l$ = unit vector in the l-direction. Equation [5] shows that the total derivative of the vector $\vec{\xi}$ with respect to time can be subdivided into an isotropic microstretch and a microrotation. The microrotation generally is not identical with the classical rotation vector.

$$\omega_r = \frac{1}{2}\,e_{rkl}\,v_{l,k}. \qquad [6]$$

An index followed by a comma means a partial differentiation with respect to space variable x_k, e. g.,

$$v_{l,k} = \frac{\partial v_l}{\partial x_k}. \qquad [7]$$

The basic laws of motion of micropolar stretch fluids are:
Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\,\mathbf{v}) = 0. \qquad [8]$$

Conservation of microinertia:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}.\nabla\right) j - 2nj = 0. \qquad [9]$$

Balance of first stress moments:

$$(\alpha + \beta)\, \nabla\nabla \cdot \mathbf{n} + \gamma\, \nabla^2 \mathbf{n} + \varkappa\, \nabla \times \mathbf{v} - 2\varkappa\, \mathbf{n} + \rho\, \mathbf{d} = \rho j \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{n}, \quad [10]$$

$$\alpha_o\, \nabla^2 n - (\eta_o - \lambda_o)\, n + \rho\, d^* = \frac{1}{2}\, \rho\, j \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) n. \quad [11]$$

Conservation of energy:

$$\begin{aligned} \rho\, c_v \left(\frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k} \right) T = &- p\, d_{kk} + \lambda_o\, n d_{kk} + \lambda\, d_{kk} d_{ll} + \\ &+ (2\mu + \varkappa)\, d_{kl} d_{lk} + 3(\eta_o - \lambda_o)\, n^2 + 3\alpha_o n_{,k} n_{,k} + \\ &+ (3\alpha_1 + \beta_o)\, e_{klr} n_{l,k} n_r + 2\varkappa(\omega_k - n_k)(\omega_k - n_k) + \\ &+ \alpha\, n_{k,k} n_{l,l} + \beta\, n_{k,l} n_{l,k} + \gamma\, n_{l,k} n_{l,k} + \varkappa^*\, T_{,kk} + Q \end{aligned} \quad [12]$$

Balance of momentum:

$$\begin{aligned} &- \nabla p + \lambda_o \nabla n + (\lambda + \mu)\, \nabla\nabla \cdot \mathbf{v} + (\mu + \varkappa)\, \nabla^2 \mathbf{v} + \varkappa\, \nabla \times \mathbf{n} + \\ &+ \rho\, \mathbf{f} = \rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v}. \end{aligned} \quad [13]$$

Equation of state:

$$\rho = \rho_o [1 - \delta\, (T - T_o)] \quad [14]$$

the classical spin tensor being denoted by d_{kl}.

$$d_{kl} = \frac{1}{2}\, (v_{k,l} + v_{l,k}) \quad [15]$$

ρ = density; T = absolute temperature; p = pressure; $\mathbf{f}$ = body force per unit mass; $\mathbf{d}$ = axial vector of the first body moments per unit mass which are closely connected with the microrotation $\mathbf{n}$; d^* = first body moment per unit mass which is closely connected with the microstretch n; c_v = specific heat at constant volume; $\varkappa^*$ = coefficient of heat conduction; Q = heat produced within the fluid per unit volume per unit time; α, α_o, α_1, β, β_o, γ, $\varkappa$, η_o, λ, λ_o, μ = viscosity constants; δ = thermal expansion coefficient; ρ_o = density at the fixed temperature T_o.

The relationships [8] to [14] are eleven scalar equations for the determination of eleven scalar unknowns: v_k, n_k, n, j, ρ, T, p. Compared with the fundamental equations of a micropolar fluid without stretch (16), n and j are *additional* sought functions. $\mathbf{f}$, $\mathbf{d}$, and d^* are to be given quantities. As Eringen (7) and Erdogan (6) have found, the material constants are subject to certain restrictions which are necessary and sufficient to ensure the validity of the principle of entropy:

$$\left.\begin{array}{lll} 3\lambda + 2\mu + \varkappa \geqslant 0; & 2\mu + \varkappa \geqslant 0; & \varkappa \geqslant 0 \\ 3\alpha + \beta\gamma \geqslant 0; & -\gamma \leqslant \beta \leqslant \gamma; & \gamma \geqslant 0 \end{array}\right\} \begin{array}{l}\text{where} \\ T \neq 0\end{array} \qquad [16]$$

and

$$\alpha_0 \geqslant 0; \quad \eta_0 - \lambda_0 \geqslant 0; \quad (\eta_0 - \lambda_0)(3\lambda + 2\mu + \varkappa) \geqslant \frac{\lambda_0^2}{4}. \qquad [17]$$

Let us now make some highly justified simplifications. The quantities $\mathbf{v}$, $\mathbf{n}$ and n are so small in the asthenosphere that quadratic and mixed quadratic terms thereof may be neglected. Therefore, the underlined expressions of the formulas [10] to [13] are neglected. The accelerations $\mathbf{d}$ and d^* are to disappear. Let, furthermore $\mathbf{f}$ be equal to $\mathbf{g}$, i. e., only the gravity acceleration is assumed to be effective. Thus, the governing equations [8] to [14] are simplified to [18] to [24]

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0, \qquad [18]$$

$$\frac{\partial j}{\partial t} + \mathbf{v} \cdot \nabla j - 2\, nj = 0, \qquad [19]$$

$$(\alpha + \beta) \nabla\nabla \cdot \mathbf{n} + \gamma \nabla^2 \mathbf{n} + \varkappa \nabla \times \mathbf{v} - 2\varkappa\, \mathbf{n} = \rho\, j \frac{\partial \mathbf{n}}{\partial t}, \qquad [20]$$

$$\alpha_0 \nabla^2 n - (\eta_0 - \lambda_0)\, n = \frac{1}{2} \rho\, j \frac{\partial n}{\partial t}, \qquad [21]$$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = - c\, p\, \nabla \cdot \mathbf{v} + k \nabla^2 T + q \qquad [22]$$

$$- \nabla p + \lambda_0 \nabla n + (\lambda + \mu) \nabla\nabla \cdot \mathbf{v} + (\mu + \varkappa) \nabla^2 \mathbf{v} + \qquad [23]$$
$$+ \varkappa \nabla \times \mathbf{n} + \rho\, \mathbf{g} = \rho \frac{\partial \mathbf{v}}{\partial t}$$

$$\rho = \rho_0 [1 - \delta(T - T_0)] \qquad [24]$$

where

$$c = \frac{1}{\rho c_v}; \qquad k = \frac{\varkappa^*}{\rho c_v}; \qquad q = \frac{Q}{\rho c_v}. \qquad [25]$$

2. – Linearized governing equations

Let us now assume that the principle of exchange of stabilities ([4]) is valid, i. e., the current is considered to be stationary at the beginning. In the marginal state at the beginning of convection the mean state is assumed to be equal to the equilibrium state, and all variables should be representable as sums of equilibrium quantities which are functions of x_3 only, and of small perturbations which are functions of x_1, x_2, x_3, and t:

$$p = \bar{p} + p'; \qquad T = \bar{T} + T'; \qquad \rho = \bar{\rho} + \rho'; \qquad [26]$$
$$\mathbf{v} = \mathbf{v}; \qquad \mathbf{n} = \mathbf{n}; \qquad n = n.$$

Mean state variables are denoted by an overbar.

Per definitionem there must not occur any motions in the equilibrium state:

$$\mathbf{v} = 0; \qquad \mathbf{n} = 0; \qquad n = 0. \qquad [27]$$

Hence, it follows from [22]:

$$\nabla^2 \bar{T} = -\frac{q}{k}; \qquad \frac{d^2}{dx_3^2}\bar{T} = -\frac{q}{k}, \qquad [28]$$

$$\bar{T} = \frac{q}{2k}(hx_3 - x_3^2) - \frac{T_0 - T_1}{h} x_3 + T_0. \qquad [29]$$

From [23] and [27]

$$\nabla \bar{p} = \bar{\rho}\, \mathbf{g}; \qquad \frac{d\bar{p}}{dx_3} = -\bar{\rho}\, g; \qquad \bar{p} = \bar{\rho}\, g(h - x_3) + p_1 \qquad [30]$$

$\mathbf{g} = -g\,\mathbf{e}_3$ being used, and the constant pressure at the upper boundary surface of the layer being denoted by p_1. From [24] and [27] we obtain

$$\bar{\rho} = \rho_0[1 - \delta(\bar{T} - T_0)]. \qquad [31]$$

Trivially, the other governing equations are satisfied by [27]. Eq. [27], [29], $[30]_3$, and [31] are the solutions for the static case.

In order to obtain the linearized fundamental equations, we now substitute [26] into the equations [18] to [27], the mean state variables being known now from the statics. We renounce here the introduction of the Boussinesq approximation usual at this point, since it is exactly here where the theory is to be extended by stretch and microstretch. We obtain:

$$\frac{\partial \rho'}{\partial t} + \mathbf{v} \cdot \nabla \bar{\rho} + \bar{\rho} \nabla \cdot \mathbf{v} = 0, \qquad [32]$$

$$\frac{\partial j}{\partial t} + \mathbf{v} \cdot \nabla j - 2 n j = 0, \qquad [33]$$

$$(\alpha + \beta) \nabla\nabla \cdot \mathbf{n} + \gamma \nabla^2 \mathbf{n} + \varkappa \nabla \times \mathbf{v} - 2\varkappa \mathbf{n} = \bar{\rho}\, j \frac{\partial \mathbf{n}}{\partial t}, \qquad [34]$$

$$\nabla^2 n = \nu^2 n + \frac{1}{2\alpha_0} \bar{\rho}\, j \frac{\partial n}{\partial t}, \qquad [35]$$

$$\frac{\partial T'}{\partial t} + \mathbf{v} \cdot \nabla \bar{T} = - c\, \bar{p} \nabla \cdot \mathbf{v} + k \nabla^2 T', \qquad [36]$$

$$- \nabla p' + \lambda_0 \nabla n + \rho' \mathbf{g} + (\lambda + \mu) \nabla\nabla \cdot \mathbf{v} + (\mu + \varkappa) \nabla^2 \mathbf{v} + \varkappa \nabla \times \mathbf{n} = \bar{\rho} \frac{\partial \mathbf{v}}{\partial t}, \qquad [37]$$

$$\rho' = - \rho_0 \delta T' \qquad [38]$$

ν^2 is defined by

$$\nu^2 = \frac{\eta_0 - \lambda_0}{\alpha_0} \qquad [39]$$

where, because of [17], $\nu^2 > 0$, if $\alpha_0 \neq 0$. In [32] and [34] to [37] all products of the small perturbation quantities p', T', ρ', $\mathbf{v}$, $\mathbf{n}$, and n have been neglected in the calculation. In [36], terms have been eliminated due to $[28]_1$ and in [37] due to $[30]_1$. *In the following we shall confine ourselves to the steady-state* case. Thus, we obtain from [32] to [38] the following system of linearized fundamental equations,

$$\mathbf{v} \cdot \nabla \bar{\rho} + \bar{\rho} \nabla \cdot \mathbf{v} = 0, \qquad [40]$$

$$\mathbf{v} \cdot \nabla j = 2 n j, \qquad [41]$$

$$(\alpha + \beta + \gamma) \nabla\nabla \cdot \mathbf{n} - \gamma \nabla \times (\nabla \times \mathbf{n}) + \varkappa \nabla \times \mathbf{v} - 2\varkappa \mathbf{n} = 0, \qquad [42]$$

$$[\nabla^2 - \nu^2]\, n = 0, \qquad [43]$$

$$k \nabla^2 T' = v_3 \frac{d}{dx_3} \bar{T} + c\, \bar{p} \nabla \cdot \mathbf{v}, \qquad [44]$$

$$\rho_0 \delta g T' \mathbf{e}_3 - \nabla p' + \lambda_0 \nabla n + (\lambda + 2\mu + \varkappa) \nabla\nabla \cdot \mathbf{v} - \qquad [45]$$
$$- (\mu + \varkappa) \text{ curl curl } \mathbf{v} + \varkappa \nabla \times \mathbf{n} = 0.$$

Equation [45] is obtained by eliminating ρ' in [37] by [38]. The equations [32] to [38] form a system of 11 partial differential equations for the determination of the eleven unknown functions $\mathbf{v}$, $\mathbf{n}$, n, j, ρ', T', p'.

3. – Reduction of the problem to an ordinary differential equation and discussion of the solution

As we already have started to do with [45], we will show in the following that the system can be decoupled step by step. To eliminate p', we take (— curl curl) of [45].

$$-\rho_0 g \delta \left(\mathbf{e}_1 \frac{\partial^2}{\partial x_1 \partial x_3} + \mathbf{e}_2 \frac{\partial^2}{\partial x_2 \partial x_3}\right) T' + \rho_0 g \delta \mathbf{e}_3 \nabla_3^2 T' + \qquad [46]$$
$$+ (\mu + \varkappa) \text{ curl curl curl curl } \mathbf{v} - \varkappa \text{ curl curl curl } \mathbf{n} = 0$$

where

$$\nabla_3^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} . \qquad [47]$$

Let us assume in the following that the divergence of the microrotation disappears.

$$\nabla \cdot \mathbf{n} = 0. \qquad [48]$$

By taking curl, we obtain from [42] and [48]

$$-\gamma \text{ curl curl curl } \mathbf{n} + \varkappa \text{ curl curl } \mathbf{v} - 2\varkappa \text{ curl } \mathbf{n} = 0, \qquad [49]$$

$$-\varkappa \text{ curl curl curl curl } \mathbf{n} = -\frac{2\varkappa^2}{\gamma} \nabla^2 \mathbf{n} - \frac{\varkappa^2}{\gamma} \text{ curl curl curl } \mathbf{v}. \qquad [50]$$

From [40] we obtain

$$-\nabla \cdot \mathbf{v} = \frac{\mathbf{v} \cdot \nabla \bar{\rho}}{\bar{\rho}} = v_3 \frac{d \ln \bar{\rho}}{dx_3} . \qquad [51]$$

From [44] and [51]

$$-k \nabla^2 T' = v_3 \left[c\, p \frac{d \ln \rho}{dx_3} - \frac{dT}{dx_3} \right] . \qquad [52]$$

As a consequence of taking curl, we obtain from [46]

$$-(\mu+\varkappa)\ \text{curl curl curl curl curl}\ \mathbf{v} = \qquad [53]$$

$$= \rho_0\, g\, \delta\, (\mathbf{e}_1\, \partial/\partial x_2 - \mathbf{e}_2\, \partial/\partial x_1)\, \nabla^2 T' - \varkappa\ \text{curl curl curl curl}\ \mathbf{n}.$$

From [50] and [53]

$$\left[-(\mu+\varkappa)\ \text{curl curl} + \frac{\varkappa^2}{\gamma}\right] \text{curl curl curl}\ \mathbf{v} = \qquad [54]$$

$$= \rho_0\, g\, \delta \left(\mathbf{e}_1 \frac{\partial}{\partial x_2} - \mathbf{e}_2 \frac{\partial}{\partial x_1}\right) \nabla^2 T' - \frac{2\varkappa^2}{\gamma}\, \nabla^2\, \mathbf{n}.$$

We apply the operator $(-k\,\nabla^2)$ to [46] and combine this equation with [52] and [48].

$$k(\mu+\varkappa)\ \text{curl curl curl curl}\ \nabla^2\, \mathbf{v} =$$

$$= \rho_0\, g\, \delta \left[\mathbf{e}_1 \frac{\partial^2}{\partial x_1\, \partial x_3} + \mathbf{e}_2 \frac{\partial^2}{\partial x_2\, \partial x_3} - \mathbf{e}_3\, \nabla_3^2\right] \qquad [55]$$

$$\left[\frac{d\bar{T}}{dx_3} - c\,\bar{p}\, \frac{d \ln \bar{\rho}}{dx_3}\right] v_3 - k\, \varkappa\, \nabla^2\, \nabla^2\ \text{curl}\ \mathbf{n}.$$

The curl of equation [54] yields

$$\left[-(\mu+\varkappa)\ \text{curl curl} + \frac{\varkappa^2}{\gamma}\right] \text{curl curl curl curl}\ \mathbf{v} = \qquad [56]$$

$$= \rho_0\, g\, \delta \left(\mathbf{e}_1 \frac{\partial^2}{\partial x_1\, \partial x_3} + \mathbf{e}_2 \frac{\partial^2}{\partial x_2\, \partial x_3} - \mathbf{e}_3\, \nabla_3^2\right) \nabla^2 T' - \frac{2\varkappa^2}{\gamma}\, \nabla^2\ \text{curl}\ \mathbf{n}.$$

From [56] and [52] we obtain

$$\left[-(\mu+\varkappa)\ \text{curl curl} + \frac{\varkappa^2}{\gamma}\right] \text{curl curl curl curl}\ \nabla^2\, \mathbf{v} = \qquad [57]$$

$$= \frac{\rho_0\, g\, \delta}{k} \left(\mathbf{e}_1 \frac{\partial^2}{\partial x_1\, \partial x_3} + \mathbf{e}_2 \frac{\partial^2}{\partial x_2\, \partial x_3} - \mathbf{e}_3\, \nabla_3^2\right) \nabla^2 \cdot$$

$$\cdot \left[\frac{d\bar{T}}{dx_3} - c\,\bar{p}\, \frac{d \ln \bar{\rho}}{d\, x_3}\right] v_3 - \frac{2\varkappa^2}{\gamma}\, \nabla^2\, \nabla^2\ \text{curl}\ \mathbf{n}.$$

We multiply equation [55] by $2\varkappa/\gamma k$ and subtract therefrom equation [57].

$$\left[(\mu + \varkappa)\ \mathrm{curl\ curl} + \varkappa\ \frac{2\mu + \varkappa}{\gamma}\right] \mathrm{curl\ curl\ curl\ curl}\ \nabla^2\ \mathbf{v} = \qquad [58]$$

$$= \frac{\rho_0\, g\, \delta}{k} \left(\mathbf{e}_1 \frac{\partial^2}{\partial x_1\, \partial x_3} + \mathbf{e}_2 \frac{\partial^2}{\partial x_2\, \partial x_3} - \mathbf{e}_3\, \nabla_{\scriptscriptstyle 3}^{\scriptscriptstyle 2}\right) \cdot$$

$$\cdot \left[\nabla^2 - \frac{2\varkappa}{\gamma}\right] \left[c\, \bar{p}\, \frac{d \ln \bar{\rho}}{dx_3} - \frac{d\bar{T}}{dx_3}\right] v_3$$

Thus, the microrotation **n** has been eliminated. Using equation [51], we now transform equation [58].

$$\left[-(\mu + \varkappa)\ \nabla^2 + \varkappa\ \frac{2\mu + \varkappa}{\gamma}\right] \nabla^2\, \nabla^2 \left[\nabla^2\, \mathbf{v} + \nabla \left(\frac{d \ln \bar{\rho}}{dx_3}\, v_3\right)\right] = \qquad [59]$$

$$= \frac{\rho_0\, g\, \delta}{k} \left(\mathbf{e}_1 \frac{\partial^2}{\partial x_1\, \partial x_3} + \mathbf{e}_2 \frac{\partial^2}{\partial x_2\, \partial x_3} - \mathbf{e}_3\, \nabla_{\scriptscriptstyle 3}^{\scriptscriptstyle 2}\right) \cdot$$

$$\cdot \left[\nabla^2 - \frac{2\varkappa}{\gamma}\right] \left[c\, \bar{p}\, \frac{d \ln \bar{\rho}}{dx_3} - \frac{d\bar{T}}{dx_3}\right] v_3.$$

The 3-component thereof is written as follows:

$$\left[-(\mu + \varkappa)\nabla^2 + \varkappa\ \frac{2\mu + \varkappa}{\gamma}\right] \nabla^2\nabla^2 \left[\nabla^2 + \left(\frac{d \ln \bar{\rho}}{dx_3}\right) \frac{\partial}{\partial x_3} + \qquad [60]\right.$$

$$\left. + \left(\frac{d^2 \ln \bar{\rho}}{dx_3^2}\right)\right] v_3 = -\frac{\rho_0\, g\, \delta}{k} \left[\nabla^2 - \frac{2\varkappa}{\gamma}\right] \left[c\, \bar{p}\, \frac{d \ln \bar{\rho}}{dx_3} - \frac{d\bar{T}}{dx_3}\right] \Delta_{\scriptscriptstyle 3}^{2}\, v_3.$$

Equation [60] serves for the determination of v_3, equation [43] for the determination of n. If j is a function of x_3 only, it follows from [41] that

$$\frac{d \ln j}{dx_3} = \frac{2n}{v_3} \qquad \text{and} \qquad j = \exp\left(\int \frac{2n}{v_3}\, dx_3\right). \qquad [61]$$

From this equation j may be calculated following the determination of v_3 and n. However, if j is a function of x_1, x_2, and x_3, the function v_1 has to be calculated from the 1-component of [59] and from v_3 which is known from the solution of [60]. Accordingly, the 2-component of [59] at a known function v_3 is a differential equation for the determination of v_2. Having, thus, determined n, v_1, v_2, and v_3 from decoupled differential equations, they are substituted into equation [41] to determine j.

We now want to bring the decisive differential equation [60] closer to the solution. From [60], [29], [30]$_3$, and [31] we obtain

$$\left[(\mu + \varkappa)\nabla^2 - \varkappa \frac{2\mu + \varkappa}{\gamma}\right] \nabla^2 \nabla^2 \left[\nabla^2 + f_1 \frac{\partial}{\partial x_3} + f_2\right] v_3 = \qquad [62]$$

$$= -\frac{\rho_0 g \delta}{k}\left[\nabla^2 - \frac{2\varkappa}{\gamma}\right] f_3 \nabla_3^2 v_3$$

where

$$f_1 = f_1(x_3) = -\delta \frac{\rho_0}{\bar{\rho}} \frac{dT}{dx_3}, \qquad [63]$$

$$f_2 = f_2(x_3) = \delta \frac{\rho_0}{\bar{\rho}}\left(\frac{q}{k} - \delta \frac{\rho_0}{\bar{\rho}}\left(\frac{dT}{dx_3}\right)^2\right), \qquad [64]$$

$$f_3 = f_3(x_3) = \left\{1 + c\,\delta\,\rho_0\left[g(h - x_3) + \frac{p_1}{\bar{\rho}}\right]\right\}\frac{dT}{dx_3} \qquad [65]$$

and

$$\bar{\rho} = \rho_0\left\{1 - \delta\left[\frac{q}{2k}(h x_3 - x_3^2) - \frac{T_0 - T_1}{h} x_3\right]\right\}, \qquad [66]$$

$$\frac{dT}{dx_3} = \frac{q}{k}\left(\frac{h}{2} - x_3\right) - \frac{T_0 - T_1}{h}. \qquad [67]$$

For the differential equation [62] we assume a separable solution.

$$v_3 = w(x_3)\, f(x_1, x_2) \qquad [68]$$

with

$$\nabla_3^2 f(x_1, x_2) + \frac{a^2}{h^2} f(x_1, x_2) = 0 \qquad [69]$$

where a is the aspect ratio of the cells. From [68] and [69] we have

$$\nabla_3^2 v_3 = -\frac{a^2}{h^2} v_3. \qquad [70]$$

We define

$$\zeta = \frac{x_3}{h} \qquad \text{and} \qquad D = \frac{\partial}{\partial \zeta}. \qquad [71]$$

From [70] and [71]

$$\nabla^2 v_3 = \left(\nabla_3^2 + \frac{\partial^2}{\partial x_3^2}\right) v_3 = \frac{1}{h^2}(D^2 - a^2)\, v_3. \qquad [72]$$

From [62]

$$\left[\nabla^2 + f_1 \frac{\partial}{\partial x_3} + f_2\right]\left[(\mu + \varkappa)\nabla^2 - \varkappa \frac{2\mu + \varkappa}{\gamma}\right]\nabla^2 \nabla^2 v_3 + \tag{73}$$
$$+ \frac{\partial v_3}{\partial x_3}\left[(\mu + \varkappa)\nabla^2 - \varkappa \frac{2\mu + \varkappa}{\gamma}\right]\nabla^2 \nabla^2 f_1 +$$
$$+ v_3\left[(\mu + \varkappa)\nabla^2 - \varkappa \frac{2\mu + \varkappa}{\gamma}\right]\nabla^2 \nabla^2 f_2 =$$
$$= - \frac{\rho_0 g \delta}{k} f_3 \left[\nabla^2 - \frac{2\varkappa}{\gamma}\right]\nabla_1^2 v_3 - \frac{\rho_0 g \delta}{k} \nabla_3^2 v_3 \left[\nabla^2 - \frac{2\varkappa}{\gamma}\right] f_3 .$$

By introducing the statement [68] into equation [73] and multiplying the resulting relationship by $h^8/(\mu + \varkappa)$, we obtain

$$[(D^2 - a^2) + f_4 D + f_5][(D^2 - a^2) - k_1](D^2 - a^2)^2 w + \tag{74}$$
$$+ f_6 D w + f_7 w = R_1 \{f_8[(D^2 - a^2) - k_2] + f_9\} a^2 w$$

where

$$f_4 = f_4(x_3) = h\, f_1(x_3), \tag{75}$$

$$f_5 = f_5(x_3) = h^2 f_2(x_3), \tag{76}$$

$$f_6 = f_6(x_3) = h\,[D^2 - k_1] D^4 f_1(x_3), \tag{77}$$

$$f_7 = f_7(x_3) = h^2 [D^2 - k_1] D^4 f_2(x_3), \tag{78}$$

$$f_8 = f_8(x_3) = \frac{h\, f_3(x_3)}{T_0 - T_1}, \tag{79}$$

$$f_9 = f_9(x_3) = \frac{h}{T_0 - T_1}[D^2 - k_2]\, f_3(x_3) \tag{80}$$

and

$$R_1 = \frac{g\, \delta\, h^3 (T_0 - T_1)\, \rho_0}{k\, (\mu + \varkappa)}, \tag{81}$$

$$k_1 = \frac{\varkappa}{\gamma} \frac{2\mu + \varkappa}{\mu + \varkappa} h^2 ; \qquad k_2 = \frac{2\varkappa}{\gamma} h^2 . \tag{82}$$

R_1, k_1, and k_2 are dimensionless constants. If $\varkappa$ disappeared, R_1 would be identical with the conventional Rayleigh number. All quantities occurring in equation [74] are dimensionless. With the derivation of equation [74] *it was possible to reduce the problem to an ordinary differential equation* of the 8th order for w as a function of ζ. Equation [74] offers a favourable starting point for a numerical solution of the problem. Eq. [74] cannot be solved analytically, because dependencies on ζ are still existing in $f_4, \ldots, f_9$. In (16) it is shown how in the special case of an *incompressible* micropolar fluid *without microstretch* and *without internal heat sources* the general solution of [74] can be analytically given and how its constants can be determined according to the boundary conditions. If only the internal heat sources disappeared, i. e., if $q = 0$, a micropolar *stretch* fluid would be described by formula [74] with the following simplified expressions for $f_4, \ldots, f_9$.

$$f_4 = [\zeta + \delta^{-1} (T_o - T_1)^{-1}]^{-1}, \qquad [83]$$

$$f_5 = -[\zeta + \delta^{-1} (T_o - T_1)^{-1}]^{-2}, \qquad [84]$$

$$f_6 = [D^2 - k_1]\, D^4 [\zeta + \delta^{-1} (T_o - T_1)^{-1}]^{-1}, \qquad [85]$$

$$f_7 = -[D^2 - k_1]\, D^4 [\zeta + \delta^{-1} (T_o - T_1)^{-1}]^{-2}, \qquad [86]$$

$$f_8 = -\left\{ 1 + c\, \delta\, \rho_o \left[gh\, (1-\zeta) + \frac{p_1}{\rho_o (1 + \delta (T_o - T_1)\, \zeta)} \right] \right\}, \qquad [87]$$

$$f_9 = -[D^2 - k_2] \left\{ 1 + c\, \delta\, \rho_o \left[gh\, (1-\zeta) + \frac{p_1}{\rho_o (1 + \delta (T_o - T_1)\, \zeta)} \right] \right\}. \qquad [88]$$

Equations [83] to [88] show that, even if we neglect internal heat sources, the solution $w(\zeta)$ cannot be analytically given for a micropolar *stretch* fluid, since the $f_4, \ldots, f_9$ explicitely contain the quantity ζ also in that case. Therefore, we may just as well solve the full problem [74] to [82] in the numerical calculations, with $q \neq 0$. The main problem, namely to reduce the system ([8] to [14]) from eleven partial differential equations with eleven unknown functions and four independent variables to an ordinary differential equation with one sought function and one independent variable, however, has been solved. Finally, we will derive the boundary conditions for w.

4. – Boundary conditions

It is obvious that for convection in the asthenosphere the case of fixed, rigid boundaries is most important, so that we will exclusively deal with this case here. The formulas [89] to [95] as well as [97] and [98] *hold only at the boundaries* $\zeta = 0$ *and* $\zeta = 1$ *of the layer.*

$$\mathbf{v} = 0; \qquad \mathbf{n} = 0; \qquad n = 0 \qquad [89]$$

Because the temperature at the boundary surfaces is to be kept constant, we have

$$T' = 0. \qquad [90]$$

From $[89]_1$

$$w = 0. \tag{91}$$

From [90] it follows that $\bar{T}$ = const; from this and from [31] it follows that $\bar{\rho}$ = const; from this, from [40], and $[89]_1$ we obtain

$$Dw = 0. \tag{92}$$

From $[89]_2$

$$\text{curl } \mathbf{n} = -\frac{\partial n_2}{\partial x_3} \mathbf{e}_1 + \frac{\partial n_1}{\partial x_3} \mathbf{e}_2 . \tag{93}$$

From [46], [48], [90], and [93]

$$0 = \mathbf{e}_3 \cdot (\text{curl curl curl curl } \mathbf{v}) = \nabla^2 \nabla^2 v_3 + \nabla^2 \frac{\partial}{\partial x_3}\left(\frac{d \ln \bar{\rho}}{dx_3} v_3\right) \tag{94}$$

Since $\dfrac{\partial v_3}{\partial x_1} = \dfrac{\partial v_3}{\partial x_2} = 0$ and $\dfrac{\partial \bar{\rho}}{\partial x_1} = \dfrac{\partial \bar{\rho}}{\partial x_2} = 0$ it follows that

$$\frac{\partial^4}{\partial x_3^4} v_3 + \frac{\partial^3}{\partial x_3^3}\left(\frac{d \ln \bar{\rho}}{dx_3} v_3\right) = 0 . \tag{95}$$

From [48] and [49] we obtain

$$\text{curl curl } \mathbf{v} = \left(2 - \frac{\gamma}{\varkappa} \nabla^2\right) \text{curl } \mathbf{n} . \tag{96}$$

From [51], [93], and [96]

$$\begin{aligned} \mathbf{e}_3 \cdot \text{curl curl } \mathbf{v} &= \mathbf{e}_3 \cdot (\nabla \nabla \cdot \mathbf{v} - \nabla^2 \mathbf{v}) = \\ &= -\mathbf{e}_3 \cdot \left[\nabla\left(v_3 \frac{d \ln \bar{\rho}}{dx_3}\right) + \nabla^2 \mathbf{v}\right] = 0, \end{aligned} \tag{97}$$

$$\frac{\partial^2}{\partial x_3^2} v_3 + \frac{\partial}{\partial x_3}\left(\frac{d \ln \bar{\rho}}{dx_3} v_3\right) = 0 . \tag{98}$$

Let us now introduce some derivatives taken at the upper and lower boundary planes

$$\left.\frac{d \ln \bar{\rho}}{dx_3}\right|_{\zeta = 0} = h^{-1} A_0 ; \qquad \left.\frac{d \ln \bar{\rho}}{dx_3}\right|_{\zeta = 1} = h^{-1} A_1 , \tag{99}$$

$$\left.\frac{d^2 \ln \bar{\rho}}{dx_3^2}\right|_{\zeta = 0} = h^{-2} B_0 ; \qquad \left.\frac{d^2 \ln \bar{\rho}}{dx_3^2}\right|_{\zeta = 1} = h^{-2} B_1 , \tag{100}$$

$$\left.\frac{d^4 \ln \bar{\rho}}{dx_3^4}\right|_{\zeta = 0} = h^{-4} C_0 ; \qquad \left.\frac{d^4 \ln \bar{\rho}}{dx_3^4}\right|_{\zeta = 1} = h^{-4} C_1 . \tag{101}$$

The quantities A_o, A_1, B_o, B_1, C_o, and C_1 defined by [99] to [101] are dimensionless constants, e. g.,

$$A_o = \delta \left\{ (T_o - T_1) - \frac{qh^2}{2k} \right\} \qquad [102]$$

and

$$A_1 = \frac{(T_o - T_1) + qh^2/(2k)}{(T_o - T_1) + \delta^{-1}} . \qquad [103]$$

By means of these six constants equations [98] and [95] can be transformed as follows.

$$D^2w + A_\zeta Dw + B_\zeta w = 0 \text{ for } \zeta = 0 \text{ and } \zeta = 1, \qquad [104]$$

$$D^4w + A_\zeta D^3w + C_\zeta w = 0 \text{ for } \zeta = 0 \text{ and } \zeta = 1. \qquad [105]$$

Thus, the boundary conditions the solution w of [74] must satisfy at the boundaries $\zeta = 0$ and $\zeta = 1$ can be summarized as follows:

$$w = Dw = (D^2 + A_\zeta D + B_\zeta)w = (D^4 + A_\zeta D^3 + C_\zeta)w = 0. \qquad [106]$$

In the special case of an incompressible micropolar fluid without internal heat sources the six constants A_ζ, B_ζ, and C_ζ disappear.

CONCLUSIONS

Whatever the principal driving mechanism for the motion of the lithospheric plates may be, it is obvious that lattice and radiative thermal diffusivities of mantle rock are not sufficient to explain the heat flux observed at the surface of the Earth. Therefore, there must be convection in the mantle and particularly in the asthenosphere. If it is assumed (1) that the low-velocity layer is partly molten and that there is a geometrical connectivity between the solid and liquid phases in the form (see Fig. 1) suggested by Stocker and Ashby (12), a layer of a micropolar stretch fluid with internal heat sources may be introduced as a model for the asthenosphere. Hence, the Bénard problem of this model has to be solved, the case of fixed, rigid boundaries being of interest. The dynamic fundamental equations form a system of eleven partial differential equations. The essential object of the present paper is to show how the equations can be decoupled and the problem be reduced to an ordinary differential equation, which is a suitable starting point for numerical calculations. In all the calculations the Boussinesq approximation has not been used and stretch and microstretch have been considered. In the special case of a micropolar fluid without stretch the solution can even be given analytically.

REFERENCES

(1) ANDERSON D. L. and SAMMIS C., 1970. – *Partial Melting of the Upper Mantle.* "Phys. Earth and Planet. Interiors", **3**, pp. 41-50.

(2) BOSCHI E., 1973. – *Lamb and Love Wave Propagation in an Infinite Micropolar Elastic Plate.* "Annali di Geofisica", **XXVI**, pp. 341-355.

(3) CASSEN P. and REYNOLDS R. T., 1974. – *Convection in the Moon: Effect of Variable Viscosity.* "J. geophys. Res.", **79**, pp. 2937-2944.

(4) CHANDRASEKHAR S., 1961. – *Hydrodynamic and Hydromagnetic Stability.* Oxford.

(5) DE BREMAECKER J. C., 1974. – *Temperatures in a Convecting Upper Mantle.* "Tectonophysics", **21**, pp. 1-13.

(6) ERDOGAN M. E., 1972. – *Dynamics of Polar Fluids.* "Acta Mechanica", **15**, pp. 233-253.

(7) ERINGEN A. C., 1969. – *Micropolar Fluids with Stretch.* "Int. Journal Engng. Sci.", **7**, pp. 115-127.

(8) KIRBY S. H. and RALEIGH C. B., 1973. – *Mechanisms of High-Temperature, Solid-State Flow in Minerals and Ceramics and Their Bearing on the Creep Behavior of the Mantle.* "Tectonophysics", **19**, pp. 165-194.

(9) McKENZIE D. and WEISS N., 1975. – *Speculations on the Thermal and Tectonic History of the Earth.* "Geophys. J. R. astr. Soc.", **42**, pp. 131-174.

(10) POST R. L., Jr. and GRIGGS D. T., 1973. – *The Earth's Mantle: Evidence of Non-Newtonian Flow.* "Science", **181**, 4106, pp. 1242-1244.

(11) RICHTER F. M. and PARSONS B., 1975. – *On the Interaction of Two Scales of Convection in the Mantle.* "J. geophys. Res.", **80**, pp. 2529-2541.

(12) STOCKER R. L. and ASHBY M. F., 1973. – *On the Rheology of the Upper Mantle.* "Rev. Geophys. and Space Phys.", **11**, pp. 391-426.

(13) TEISSEYRE R., 1973. – *Earthquake Processes in a Micromorphic Continuum.* "Pure and Appl. Geophysics", **102** I, pp. 15-28.

(14) TURCOTTE D. L., TORRANCE K. E. and HSUI A. T., 1973. – *Convection in the Earth's Mantle.* "Methods in Computational Physics", **13**, pp. 431-454.

(15) WALZER U., 1973. – *A Quantitative Kinematic Theory of Convection Currents in the Earth's Mantle.* "Pure and Appl. Geophysics", **105**, IV, pp. 669-695.

(16) WALZER U., 1976. – *Convection of a Cosserat Fluid.* "Gerlands Beitr. Geophysik", **85**, 2, pp. 137-150.

(17) WALZER U., 1977. – *A Theory of Non-Newtonian Mantle Convection.* "Gerlands Beitr. Geophysik", **86**, 1, pp. 51-61.

(18) WEERTMAN J., 1970. – *The Creep Strength of the Earth's Mantle.* "Rev. Geophys. Space Phys.", **8**, pp. 145-168.